Finite Geometry and Graph Theory Intertwine: Turán Numbers

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Promotor: Prof. dr. L. Storme

Master thesis submitted to obtain the academic degree of Master in Mathematics, specialization Pure Mathematics.
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Introduction

Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians.

Béla Bollobás

Extremal graph theory, and most of combinatorics for that matter, are relatively recent subjects in mathematics, compared to for example analysis or number theory. It has seen a lot of growth during the 20th century, mainly under the impulse of Erdős and other Hungarian mathematicians, as Bollobás also remarked. In this thesis we will discuss a subject in extremal graph theory which originates from a theorem by another Hungarian mathematician, Pál Turán. In 1941 he investigated the following problem: what is the maximal number of edges a graph on \( n \) vertices can have, without containing a complete graph \( K_r \) ? This result was the genesis of a subfield of extremal graph theory which is nowadays called Turán problems. In this thesis, we will focus on the following problem: what is the maximal number of edges a graph on \( n \) vertices can have, without containing a fixed graph \( H \)? This problem is also known as the forbidden subgraph problem. The maximum is denoted as \( \text{ex}(n, H) \) and studied as a function of \( n \). When exact values are known of this function for certain \( n \), one then tries to investigate the extremal graphs, which are denoted as \( \text{EX}(n, H) \).

When \( H \) is not bipartite, Erdős, Stone and Simonovits determined the asymptotic growth of \( \text{ex}(n, H) \), this is called the non-degenerate case. In the degenerate case, when \( H \) is bipartite, there is not much known. This is the case on which we will focus most. The general procedure to investigate \( \text{ex}(n, H) \) in the degenerate case is twofold: on the one hand, one tries to find upper bounds, based on graph-theoretical arguments. On the other hand, using for example finite geometries, one tries to find matching lower bounds. This is where the interplay between graph theory and finite geometry will appear most clearly. In Chapter 1 we investigate the function \( \text{ex}(n, H) \) for arbitrary \( H \) and look at a few instances where the values of these functions are known for all \( n \). This chapter serves as an introduction to the field and develops the necessary graph theory. In Chapter 2 we look at the case when \( H \) is the quadrilateral \( C_4 = K_{2,2} \). For certain values of \( n \), \( \text{ex}(n, C_4) \) and \( \text{EX}(n, C_4) \) have been determined. We will see that finite projective planes play a big role here and appear naturally when considering \( C_4 \)-free graphs. In Chapter 3 we generalise this to general complete bipartite graphs \( K_{s,t} \). We will see that designs come into play here. It is no coincidence that projective planes are a subclass of designs. Lastly, in Chapter 4, we generalise \( C_4 \) to even cycles \( C_{2k} \). Again, there exists finite geometries that lead to good constructions of \( C_{2k} \)-free graphs. These are the generalized polygons, which also have projective planes as a subclass. What we will see in these three chapters is that finite geometries lend themselves very well to constructing infinite families of graphs satisfying certain properties.

The chapters themselves have following structure: first we introduce some theory which will play a big role in the chapter. In our four chapters, we introduce graph theory, projective planes and spaces, designs and generalized polygons respectively. These are mainly written for the reader that is not familiar with these subjects. Veterans in the field will probably be able to quickly skim through these parts, briefly recalling some definitions and concepts in the process. Chapter 1 is a bit of an outsider, as this one does not introduce any finite geometry, and serves mainly as an introduction to graph theory. The other chapters first develop some theory from finite geometry. Then we investigate the known
upper bounds, but the most important part is finding good lower bounds. This is where the theory we
develop at the beginning comes into play. We will investigate the opportunities and limits of the finite
geometry constructions and end each chapter with some open questions.

We will not always mention this explicitly, but every object we work with is finite. Moreover, we
make following conventions. The notation $\mathbb{N}$ is used for the non-negative integers. Unless explicity
stated otherwise, arabic letters will denote non-negative integers, i.e., when we write $k$ or $k \geq 1$, we
assume $k \in \mathbb{N}$. On the other hand $c > 0$ denotes a non-negative real number and $\epsilon > 0$ is the usual
notation for a small non-negative real number. A set of only one element will often be written without
accolades. If $A$ is a set, then $|A|$ denotes its size and $\binom{A}{k}$ all subsets of $A$ having size $k$. If we say ‘take
$k$ vertices’, we mean we take $k$ distinct vertices. In the same vein, $x, y \in A$ will imply that $x \neq y$.
Lastly, we use following asymptotic notation. Let $f(n)$ and $g(n)$ be functions from $\mathbb{N}$ to $\mathbb{N}$, then we
have following asymptotic notation.

- $f(n) = O(g(n))$ if and only if $f(n) \leq cg(n)$, where $c > 0$, for $n$ sufficiently large,
- $f(n) = o(g(n))$ if and only if $g(n) \neq 0$ and $\lim_{n \to \infty} f(n)/g(n) = 0$,
- $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $g(n) = O(f(n))$,
- $f(n) \approx g(n)$ if and only if $\lim_{n \to \infty} f(n)/g(n) = 1$,
- $f(n) = \Omega(g(n))$ if and only if there are infinitely many $n$ such that $cg(n) \leq f(n)$ for some
  $c > 0$.

Using this notation, we can write results very compact. On the other hand, it is not very useful
for proofs. If we have to proof a statement containing these notations, we will resort to standard
arguments containing $\epsilon, \delta, n_0$, making computations easier.

As every other author, we have to make decisions what to include and what to omit. Computational-
heavy, long or technical proofs are generally not included. We hope that we have found a good balance
between the two.

I thank Leo Storme, my promotor, for suggesting this subject. It has been a real treat to work on.
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Sam Matheus
Gent, May 31, 2016
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Chapter 1

Forbidding subgraphs

In this chapter we will give an introduction to the earliest and most fundamental results of this branch of extremal graph theory. We start off with some basic graph theory, most of which can also be found in Diestel’s standard work [23]. We cannot demand from that the reader that he remembers every definition, notation and concept we talk about here. Therefore, we will develop most of the general definitions and notations, but define some additional properties and concepts in the appropriate chapters and sections.

1.0 Foundations: graph theory

1.0.1 Definitions and notation

A graph $G$ consists of a pair of finite sets $(V, E)$: a non-empty set of vertices $V$ and a set of edges $E \subseteq \binom{V}{2}$, which, as we recall, are subsets of $V$ of size two. For some authors this means that the graph is finite and simple, i.e. without loops or multiple edges. While we do not allow $V$ to be empty, $E$ can be. If this is the case, then $G$ is the empty graph $\emptyset$. The vertex set of $G$ is denoted by $V(G)$ and its edge set by $E(G)$. This notation is independent of the names of the vertex set and edge set of $G$, i.e., if $G = (U, F)$ then $V(G) = U$ and $E(G) = F$. Their respective sizes are denoted by $v(G)$ and $e(G)$.

We denote an edge $\{v, w\}$ also by $vw$. If $vw$ is an edge, then $v$ and $w$ are called the endpoints of this edge. The complement of a graph $G = (V, E)$ is $\overline{G} = (V, \binom{V}{2} \setminus E)$.

Two graphs $G$ and $G'$ are isomorphic if there exists a bijection $\phi$ between $V(G)$ and $V(G')$ such that $vw \in E(G)$ if and only if $\phi(v)\phi(w) \in E(G')$. In general, one does not differentiate between isomorphic graphs as they are indistinguishable as combinatorial structures. When making figures, we draw vertices as dots and edges as lines connecting the dots; an example is the graph on the cover. Because we do not differentiate between isomorphic graphs, it does not matter how the edges are drawn, only which dots are connected and which are not.

Two vertices $v, w \in V(G)$ are adjacent when $vw \in E(G)$, we denote this by $v \sim w$. Clearly if $v \sim w$ then also $w \sim v$. If $C \subseteq V(G)$ is a set of mutually adjacent vertices, then $C$ is a clique. If $I \subseteq V(G)$ is a set of mutually non-adjacent vertices, then $I$ is an independent set or coclique, and the vertices are said to be independent. The neighbourhood $N(v)$ of a vertex $v \in V(G)$ is the set $\{w \in V(G) \mid v \sim w\}$, the degree of a vertex $v$ is $d(v) = |N(v)|$. By counting in two ways the endpoints of edges one can see that

$$2e(G) = \sum_{i=1}^{n} d(v_i).$$
Foundations: graph theory

where \( V(G) = \{v_1 \ldots, v_n\} \). The minimal and maximum degree of \( G \) are denoted by \( \delta(G) \) and \( \Delta(G) \) respectively, i.e.

\[
\delta(G) = \min_{v \in V(G)} d(v), \quad \Delta(G) = \max_{v \in V(G)} d(v).
\]

We give an example of these concepts in the following figure.

![Figure 1.1: In red: a clique of size 4, \( K_4 \).
The green: an independent set, or coclique, of size 3.
The in blue: vertices of minimum (dark) and maximum degree (light).](image)

If in particular, \( \delta(G) = \Delta(G) = k \) then \( G \) is called k-regular or regular when \( k \) is clear from the context or unknown. The graph in Figure 1.1 is clearly not regular. The empty graph is 0-regular.

**Proposition 1.1.** A k-regular graph on \( n \) vertices exists if and only if \( n \geq k + 1 \) and \( kn \) is even.

**Proof.** Suppose a k-regular graph \( G \) on \( n \) vertices exists, then

\[
2e(G) = \sum_{i=1}^{n} d(v_i) = nk,
\]

where again, \( V(G) = \{v_1 \ldots, v_n\} \). From this it follows that \( nk \) should be even. The condition \( n \geq k + 1 \) follows from the fact that \( G \) should contain \( v_1 \) and \( N(v_1) \), which are \( k + 1 \) vertices in total.

Now assume that \( n \geq k + 1 \) and \( kn \) is even. Then we construct a k-regular graph on \( n \) vertices in the following way: the vertices are \( \{0, \ldots, n-1\} \). If \( k = 2m \) then the vertex \( i \) is adjacent to \( i - m, i - m + 1, \ldots, i - 1, i + 1, \ldots, i + m \), modulo \( n \). If \( k = 2m + 1 \), and so \( n \) is even, then \( i \) is adjacent to \( i - m, i - m + 1, \ldots, i - 1, i + 1, \ldots, i + m \) and \( i + n/2 \), all modulo \( n \). As \( n \geq k + 1 \), this construction is well-defined, so all the \( k \) vertices adjacent to the vertex \( i \) are different. \( \square \)

The bound \( n = k + 1 \) can be reached as we will see later. Remark that if \( G \) is \( k \)-regular, then \( \overline{G} \) is \( (n-k-1) \)-regular.

Let \( G \) and \( H \) be two graphs. If there exists an injective map \( \phi \) from \( V(H) \) to \( V(G) \) such that \( \phi(v)\phi(w) \in E(G) \) if \( vw \in E(H) \), then \( H \) is a subgraph of \( G \). We also say that \( G \) contains \( H \) or \( H \) is contained in \( G \) and it is denoted as \( H \subseteq G \). If \( G \) does not contain \( H \) as a subgraph, then \( G \) is \( H \)-free. If we replace the last condition in the definition of subgraph by \( \phi(v)\phi(w) \in E(G) \) if and only if \( vw \in E(H) \), then \( H \) is an induced subgraph. Intuitively, \( H \) is isomorphic to a piece
of $G$, or to be more precise, we can obtain $H$ from $G$ by deleting vertices from $G$. By deleting a set of vertices $D \subseteq V$ from a graph $G = (V, E)$, we mean deleting the vertices in $D$ and all edges containing a deleted vertex. The graph without the deleted vertices will be denoted by $G - D$ and is equal to $(V \setminus D, E \cap (V \setminus D)^2)$. If $D = \{v\}$, we write $G - v$ instead of $G - \{v\}$. Similarly, we can delete a subgraph $H$ of $G$ by considering $G - V(H)$, we will also denote this as $G - H$. An example is given in Figure 1.2 where we delete a set of vertices from the graph in Figure 1.1.

![Figure 1.2: Deleting the red vertices from the graph.](image)

Let $G = (V, E)$ be a graph. If $W \subseteq V$ is a subset of vertices then the subgraph induced by $W$ is $(W, (W)^2 \cap E)$ and $e(W) = |(W)^2 \cap E|$. The complement of a subgraph $H \subseteq G$ is the subgraph induced by $V \setminus V(H)$, the complement of $V(H)$. Using these definitions, we can see that if $D \subseteq V$, then $G - D$ is the complement of the subgraph induced by $D$. If $A, B \subseteq V$ are two subsets of vertices then $E(A, B) = \{vw \in E \mid v \in A, w \in B\}$ and $e(A, B)$ is the size of this set. These are all the edges between vertices in $A$ and $B$, contained in $G$.

### 1.0.2 Examples of graphs

We now introduce some infinite families of graphs. First, we use the following convention for the remainder of the thesis: if a graph receives the subscript $n$, it means that this graph has $n$ vertices. The complete graph $K_n$ has $E(K_n) = (V(K_n))^2$ or equivalently, it is the complement of the empty graph. It is $(n - 1)$-regular (and therefore the bound $n = k + 1$ for $k$-regular graphs can be reached as we remarked earlier). The complete graph $K_n$ contains every other graph on at most $n$ vertices and has the most edges out of every graph on at most $n$ vertices.

The cycle $C_n$, $n \geq 3$, has $V(C_n) = \{v_1, \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} \mid i = 1, \ldots, n\}$, where the subscripts run modulo $n$. If $n$ is even, respectively odd, then $C_n$ is an even, respectively odd cycle. Remark that $K_3 = C_3$.

The path $P_n$, $n \geq 2$, has $V(P_n) = \{v_1, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} \mid i = 1, \ldots, n-1\}$. The vertices $v_1$ and $v_n$ are commonly referred to as the endpoints of the path $P_n$. The length of the path $P_n$ is $n - 1$, this is the number of edges we have to pass along this path to go from $v_1$ to $v_n$.

Let $V_1$ and $V_2$ be two disjoint non-empty sets such that $|V_1| = a$ and $|V_2| = b$. The complete bipartite graph $K_{a,b}$ is defined in the following way. Its vertex set is $V = V_1 \cup V_2$ and its edge set $E = \{vw \mid v \in V_1, w \in V_2\}$. In other words, we have every edge between $V_1$ and $V_2$, but $V_1$ and $V_2$ are independent sets themselves. In an analogous way, if $V_1, \ldots, V_k$ are pairwise disjoint sets such that $|V_i| = n_i$, then the complete $k$-partite $K_{n_1, \ldots, n_k}$ has as vertex set the union of all $V_i$ and the set \(\{vw \mid v \in V_i, w \in V_j, i \neq j\}\) as edge set. As a trivial example, $K_{1,1,1} = K_3$.

---

1. Beware that our definition of a path might be a bit different from other authors. Many times $P_n$ will indicate a path on $n + 1$ vertices and $n$ edges. However, we want to stick to our convention about subscripts and therefore in our definition, $P_n$ has $n$ vertices $n - 1$ edges.
1.0.3 More graph theoretic properties

With these families of graphs to our disposal, we can define the standard notions of connectedness and the chromatic number of a graph.

Let \( G \) be a graph. If for every two vertices \( v, w \in V(G) \) there exists a path \( P_k \subseteq G \) such that \( v, w \) are the endpoints of \( P_k \), then \( G \) is connected. For example, \( K_n \), \( C_n \) and \( P_n \) are connected graphs. Notice that also the graph consisting of a single vertex is connected by our definition. For any graph \( G \), we can define the connected components of \( G \) as its connected subgraphs which are not properly contained in any other connected subgraph of \( G \). Remark that if \( G \) is connected, its only connected component is \( G \) itself. If \( G \) is a connected graph and there exists a set of vertices \( S \), such that upon deleting them \( G - S \) is not connected, then \( S \) is a cut-set, as it ‘cuts’ \( G \) into several connected components. For example in Figure 1.2, the red vertices are a cut-set. If \( \{v\} \) is a cut-set then \( v \) is a cut-vertex. For any connected graph that is not the complete graph, there exists a cut-set \( S \). For example, take any two non-adjacent vertices \( v, w \in V(G) \), then \( S = V(G) \setminus \{v, w\} \) is a cut-set. A graph \( G \) is called \( k \)-connected if \( v(G) \geq k + 1 \) and there exist no cut-sets of size \( k \). We see that \( C_n \) is 2-connected and \( P_n \) is 1-connected. Remark that if a graph is \( k \)-connected, \( k \geq 2 \), then it is also \((k - 1)\)-connected.

If \( G \), not necessarily connected, does not contain any cycles, then \( G \) is called a forest and its connected components are called trees. Therefore, as we remarked earlier, a connected graph containing no cycles is a tree. For example, the empty graph on \( n \geq 2 \) vertices and the path are a forest and a tree respectively. Another example of a tree is that of the star \( S_n, n \geq 2 \). It consists of one vertex that is adjacent to all other vertices and nothing more: \( V(S_n) = \{v_1, \ldots, v_n\} \) and \( E(S_n) = \{v_1v_i \mid i = 2, \ldots, n\} \). It is clear that \( S_2 = P_2 \) and \( S_n = K_{1,n-1} \). We see that every tree we encountered so far satisfies \( e(G) = v(G) - 1 \). One can show that this is also a sufficient criterium to conclude that a connected graph is a tree.

Let \( G \) be a graph. The chromatic number \( \chi(G) \) of \( G \) is the minimal \( k \) such that there exist \( n_1, \ldots, n_k \) for which \( G \subseteq K_{n_1, \ldots, n_k} \). If \( G \) is not the empty graph, then clearly \( k \geq 2 \). In general, the chromatic number is defined in terms of colourings of vertices, this provides another way of looking at the problem. In a nutshell, the chromatic number of a graph is the smallest number of colours one can use to colour the vertices such that no two adjacent vertices share the same colour. If \( \chi(G) = k \), the \( k \) independent sets in which we partition the set of vertices \( G \) (following our original definition) are often called its (chromatic) classes.

Example 1.2. The complete graph \( K_n \) has chromatic number \( \chi(K_n) = n \). For the path we see that \( \chi(P_k) = 2 \) for all \( k \geq 2 \). The cycles fall in two cases, depending on the parity of \( n \): \( \chi(C_{2k}) = 2 \) and \( \chi(C_{2k+1}) = 3 \). Clearly \( \chi(K_{n_1, \ldots, n_k}) = k \).

Graphs for which \( \chi(G) = k \) are called \( k \)-partite. In particular when \( k = 2 \), we say that the graph is bipartite. Obviously, the complete bipartite \((k\text{-partite})\) graph is indeed bipartite \((k\text{-partite})\) by this
Forbidding subgraphs

definition. Remark that no \((k + 1)\)-partite graph can be contained in a \(k\)-partite graph. As a special case, we see that the complete bipartite graph \(K_{a,b}\) contains no odd cycles \(C_{2k+1}\) for all \(k \geq 1\).

1.1 Genesis

Now that we have the necessary background in graph theory, we can take a first look at the branch of extremal graph theory which had its first result by Mantel in 1907 [56], but really started to blossom after the result of Turán in 1941 [76]. We will prove both of these results with multiple proofs, each with its own distinct flavour. There are many more proofs than the ones we show here but we have picked out the most elegant ones.

Since we are in the branch of extremal graph theory, we expect to minimise or maximise some graph parameters under certain conditions. The central question of this particular field is:

for a fixed graph \(H\), what is the maximum number of edges in an \(H\)-free graph on \(n\) vertices?

Introducing the notation \(\text{ex}(n,H)\), which will be the most important function throughout this thesis, we can write

\[
\text{ex}(n,H) = \max \{ e(G) \mid v(G) = n, \ H \nsubseteq G \}.
\]

We remark that this definition can be generalised by replacing the graph \(H\) by a (finite) family of graphs. Most of the theorems in this chapter are still valid when replacing \(H\) by a finite family of graphs and adapting the result slightly, but we will only consider the case when the family consists of one graph \(H\) anyway. We refer to the excellent survey of Füredi and Simonovits for these more general results [41].

There are some easy consequences to this definition. First, if \(L \subseteq H\) then it is immediate that \(\text{ex}(n,L) \leq \text{ex}(n,H)\). Second, if a graph \(G\) on \(n\) vertices has more than \(\text{ex}(n,H)\) edges, it contains \(H\) as a subgraph. This is an alternative way of looking at the problem and actually a technique to prove upper bounds on \(\text{ex}(n,H)\): take any graph on \(n\) vertices and \(m + 1\) edges. If it always contains \(H\) as a subgraph, then \(\text{ex}(n,H) \leq m\). We will use this technique throughout. Note that it is possible for an \(H\)-free graph on \(n\) vertices and \(m < \text{ex}(n,H)\) edges to contain \(H\) when only one edge is added. Therefore it is important to consider any graph on \(n\) vertices and \(m\) edges when proving an upper bound.

Example 1.3. The graph \(C_5\) is \(K_3\)-free but does not have \(\text{ex}(5,K_3)\) edges, as we will show in Theorem 1.4. Adding any edge however, produces a \(K_3\).

Generally for fixed \(H\), computing the function \(\text{ex}(n,H)\) exactly for all \(n\) is a hard thing to do. The general practice to study such functions in extremal graph theory is the following.

1. Find the order of magnitude of the function.
2. Find the correct constant accompanying this main term.
3. Determine the lower order terms (optional).
4. Find exact values for certain \(n\).

Removing some ambiguity of the English language: 'more' means at least one higher for us. Similarly, 'less' means at least one lower.
In this thesis, the **order of magnitude** of a function \( f(n) \) is \( \alpha \) such that \( f(n) = \Theta(n^\alpha) \), i.e, we will only encounter polynomials with rational exponents. In the case \( f(n) = \text{ex}(n, H) \), \( f(n) = O(n^2) \) always holds, so \( \alpha \leq 2 \).

It is characteristic in extremal graph theory to find theorems of the form ‘for \( n \) big enough, the following holds …’. Heuristically, this is due to the fact that for larger \( n \), we can find more structure in these large graphs. Exceptional examples occur faster when the graphs we deal with are small. This does not only happen in combinatorics, think for example of the Central Limit Theorem in statistics. This is one of the reasons it is often easier to study the asymptotic behaviour of functions such as \( \text{ex}(n, H) \) and explains why this is first on the list. We will show explicit examples of this phenomenon already in this chapter.

Suppose now that we can in fact compute \( \text{ex}(n, H) \) exactly for certain \( n \). A natural next question would be to ask what \( H \)-free graphs on \( n \) vertices reach this bound. We will denote this set of graphs by \( \text{EX}(n, H) \):

\[
\text{EX}(n, H) = \{ G \mid v(G) = n, e(G) = \text{ex}(n, H), H \not\subseteq G \}.
\]

If \( G \in \text{EX}(n, H) \) we call \( G \) an **\( H \)-extremal graph** or just an **extremal graph** if \( H \) is clear from the context. These two notations together allow us to write theorems very compactly as we will see in the following few pages, starting with Mantel’s result from 1907 \[56\].

**Theorem 1.4.** \( \text{ex}(n, K_3) = \lfloor n^2/4 \rfloor \) and moreover, \( \text{EX}(n, K_3) = K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor} \).

Returning to Example \[13\], it is now clear that \( C_5 \) does not have \( \text{ex}(5, K_3) = 6 > 5 \) edges. We will prove Theorem \[1,4\] in two different ways. For the first proof we need the following easy inequality. It is a particular instance of the arithmetic mean / geometric mean inequality.

**Lemma 1.5.** Let \( a, b \in \mathbb{N} \), then \( ab \leq \left( \frac{a + b}{2} \right)^2 \) and equality holds if and only if \( a = b \).

**Proof.** This is readily shown: for \( a, b \in \mathbb{N} \) we have \((a - b)^2 \geq 0\) with equality if and only if \( a = b \). Expanding this and adding \( 4ab \) to both sides we find

\[
a^2 + 2ab + b^2 \geq 4ab,
\]

from which the full lemma is immediate. \( \square \)

**First proof of Theorem \[1,4\]** Let \( G = (V, E) \) be a triangle-free graph, \( A \) its largest independent set and \( B = V \setminus A \) its complement. Since for all \( x \in V \) we have that \( N(x) \) is an independent set, we have \( d(v) \leq |A| \). Furthermore, every edge has at least one of its endpoints in \( B \). Putting this together we find

\[
e(G) \leq \sum_{v \in B} d(v) \leq |A||B| \leq \left( \frac{|A| + |B|}{2} \right)^2 = \frac{n^2}{4},
\]

where equalities hold when the subgraph induced by \( B \) is the empty graph and \( d(v) = |A| \) for all \( v \in V \) for the first and second inequalities respectively. Now, when \( n \) is even, we have equality in the last part when \( |A| = |B| = n/2 \) and hence \( G = K_{n/2, n/2} \). When \( n \) is odd, the expression \( |A||B| \) is maximised when \( |A| \) is \( \lfloor n/2 \rfloor \) or \( \lceil n/2 \rceil \), both possibilities lead to \( G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil} \). \( \square \)

For the second proof of Theorem \[1,4\] we have to recall a special case of the Cauchy-Schwarz inequality. Again, this inequality can be formulated more generally than this tailor-made case, but this is unnecessary for our goal.

**Lemma 1.6.** Let \( \{d_1, \ldots, d_n\} \subseteq \mathbb{N} \), then

\[
\frac{1}{n} \left( \sum_{i=1}^{n} d_i \right)^2 \leq \sum_{i=1}^{n} d_i^2,
\]

with equality if and only if all \( d_i \) are equal.
Proof. This follows from the following calculations:

\[
0 \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (d_i - d_j)^2
= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (d_i^2 - 2d_i d_j + d_j^2)
= n \sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} d_i d_j
= n \sum_{i=1}^{n} d_i^2 - \left( \sum_{i=1}^{n} d_i \right)^2.
\]

By the first line we have that the equality holds if and only if all \(d_i\) are equal. \(\square\)

Second proof of Theorem 1.4. Let \(G = (V, E)\) be a triangle-free graph on \(n\) vertices and \(xy\) an edge. We know that \(N(x)\) and \(N(y)\) are disjoint, since otherwise we would have a triangle, hence \(n \geq d(x) + d(y)\). Now summing over all edges we find

\[
n \cdot e(G) = \sum_{uv \in E} (d(u) + d(v)) = \sum_{v \in V} d(v)^2,
\]

where the second equality is due to the fact that every \(v \in V\) appears \(d(v)\) times in the first sum. By the previous lemma we have that

\[
\sum_{v \in V} d(v)^2 \geq \frac{1}{n} \left( \sum_{v \in V} d(v) \right)^2 = \frac{4(e(G))^2}{n}.
\]

It follows that \(e(G) \leq n^2/4\).

The statement about \(\text{EX}(n, K_{3})\) can be done in the same way as in the first proof. \(\square\)

This concludes our proofs of the result by Mantel. In 1941, this was generalised by Turán from \(K_3\) to \(K_{r+1}\) for any \(r \geq 2\) (the case \(r = 1\) is trivial) \[76\]. The reason we use the notation \(r + 1\) instead of \(r\) is mostly aesthetically, as will become clear later on.

We can find a lower bound for \(\text{ex}(n, K_{r+1})\) by constructing the following \(K_{r+1}\)-free graph.

Construction 1.7. Consider a complete \(r\)-partite graph on \(n\) vertices, \(1 \leq r \leq n\), with its class sizes as equal as possible. This means that if \(n_1, \ldots, n_r\) are the sizes of the classes, then \(|n_i - n_j| \leq 1\) for all \(1 \leq i, j \leq r\). When \(r \geq n\) we could also construct this graph, but it is clear that it is then just \(K_n\) for any such \(r\).

If \(r\) divides \(n\), then the number of edges in this graph, which is called the Turán graph and denoted by \(T_{n,r}\), can be easily computed and equals

\[
e(T_{n,r}) = \frac{1}{2} \left( 1 - \frac{1}{r} \right) n^2.
\]

If \(r\) does not divide \(n\), then we can find \(s\) such that \(r\) divides \(n - s\) and \(0 < s \leq r - 1\). Then by the previous arguments we have

\[
e(T_{n-s,r}) = \frac{1}{2} \left( 1 - \frac{1}{r} \right) (n-s)^2 \\
\geq \frac{1}{2} \left( 1 - \frac{1}{r} - \epsilon \right) n^2,
\]
for some $\epsilon > 0$ when $n$ is large enough.
The graph $T_{n,r}$ indeed cannot contain $K_{r+1}$ since the former can be coloured with $r$ colours while
the latter cannot and therefore it is $K_{r+1}$-free. Turán’s theorem will show that is actually the unique
extremal graph for $K_{r+1}$.
Remark that for $s \leq r$ we have $e(T_{n,s}) \leq e(T_{n,r})$. In particular, when $r \geq n$ we find the complete
graph $K_n = T_{n,r}$, which of course has the most edges among all graphs on $n$ vertices. We will use this
in the proof of the theorem.

**Theorem 1.8.** $\text{ex}(n, K_{r+1}) \leq \frac{1}{2} \left( 1 - \frac{1}{r} \right) n^2$ for all $r \geq 2$ and equality holds if and only if $r$ divides $n$.
Moreover, $\text{EX}(n, K_{r+1}) = T_{n,r}$ for any $n \geq 1$.

For $r = 1$, the theorem is rather vacuous, therefore we omit it in the statement.

**First proof of Theorem 1.8** We will use induction on $n$. For $n = 1, \ldots, r$, we have that $n \leq r$ and hence

$$e(T_{n,r}) = e(K_n) = \frac{1}{2} \left( 1 - \frac{1}{n} \right) n^2 \leq \frac{1}{2} \left( 1 - \frac{1}{r} \right) n^2,$$

with equality only if $n = r$.
So assume it is correct for $n - 1$. Let $G = (V, E) \in \text{EX}(n, K_{r+1})$, then $G$ contains the complete graph
$K_r$ (otherwise we could add edges). Denote this subgraph by $A$ and let $B$ be its complement, which
has $v(B) = n - r$. Since $B$ contains no $K_{r+1}$, we have by induction that $e(B) \leq \frac{1}{2} (1 - \frac{1}{r}) (n - r)^2$.
Furthermore, every vertex in $B$ can have at most $r - 1$ neighbours in $A$. Summing all of these edges we get

$$e(A) + e(A, B) + e(B) \leq \frac{r(r - 1)}{2} + (n - r)(r - 1) + \frac{1}{2} \left( 1 - \frac{1}{r} \right) (n - r)^2$$

$$= \frac{1}{2} \left( 1 - \frac{1}{r} \right) n^2,$$

with equality only if $r$ divides $n$.
To find the extremal graphs, we see that for $n = 1, \ldots, r$, the extremal graph is $K_n = T_{n,r}$. To have
equality in (1.1), it is clear that our graph $G$ should consist of a complete graph $A$, the Turán graph
$T(n-r),r = B$ by induction and every vertex in $B$ has $r - 1$ neighbours in $A$. Now we will be able to
construct $T_{n,r}$ from this decomposition. Given a vertex $x$ in a class $X$ in $B$, we know that there is a
unique vertex $z$ in $A$ such that $xz \notin E(G)$. Now take a vertex $y$ in $B \setminus X$. We claim that $yz \in E(G)$.
Suppose $yz \notin E(G)$, then $x$ and $y$ would have the same $r - 1$ neighbours in $A$ and are adjacent since
they are in different classes, hence we found a $K_{r+1}$, a contradiction. Varying $y$, we see that every
vertex not in $X$ is adjacent to $z$. Moreover, if two vertices are in different classes, then they are non-
adjacent to different vertices in $A$. As any $x' \in X$ has to have $r - 1$ neighbours in $A$, it follows that
$x'$ and $z$ are not adjacent. In this way, one can see that every class in $B$ has a unique vertex in $A$ to
which none of its vertices are adjacent to. Then for every class, add its unique corresponding vertex in
$A$ to the class, one can see that the resulting graph is $T_{n,r}$.

The second proof of this result is one straight from the Book.

**Second proof of Theorem 1.8** Let $G \in \text{EX}(n, K_{r+1})$. We claim that non-adjacency is an equivalence
relation on $V(G)$. Given this fact, we can divide $V(G)$ into $k$ equivalence classes, it follows that $G$ is a
complete $k$-partite graph and hence $k \leq r$. Also, the classes should be as even as possible, for if there
existed classes $X$ and $Y$ such that $|X| > |Y| + 1$, we could move one vertex from $X$ to $Y$. Doing this
creates $|X| - 1$ edges and deletes $|Y|$ edges, ultimately we gain $|X| - 1 - |Y| > 0$ edges. Therefore

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3. Paul Erdős often talked about The Book, in which the perfect proofs for mathematical theorems are compiled, they shine in
their ingenuity, elegance or cleverness. Aigner and Ziegler have compiled some of these proofs in *Proofs from the BOOK*. 

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8
For any graph $G = T_{n,k}$ but as we saw in Construction 1.7, this is only maximal when $k = r$.

It remains to prove the claim. The relation is clearly reflexive and symmetric. To see that it is also transitive, take vertices $x, y, z \in V$ such that $xz, yz \notin E(G)$ and assume $xy \in E(G)$. If $d(z) < d(x)$, we can delete $z$ and create a new vertex $x'$ with $N(x') = N(x)$, call this new graph $G'$. This graph is also $K_{r+1}$-free: suppose it contains a $K_{r+1}$, then this subgraph has to contain $x'$ (otherwise $G$ already contained a $K_{r+1}$) and so it cannot contain $x$ (since $xx' \notin E(G')$). But then replacing the role of $x'$ by $x$, which has the same neighbours, we find a $K_{r+1}$ in $G$, which is a contradiction. Finally, this graph has $e(G') = e(G) - d(z) + d(x) > e(G)$ edges, this contradicts the maximality of $G$.

Therefore, we can assume $d(z) \geq d(x)$. Replacing $x$ by $y$ and repeating this reasoning, we can furthermore assume that $d(z) \geq d(y)$. Now we delete $x$ and $y$ and create vertices $z', z''$ with $N(z') = N(z'')$. This new graph $G'$ is again $K_{r+1}$-free (by the same argument as before) and has $e(G') = e(G) + 2d(z) - d(x) - d(y) + 1 > e(G)$ edges, which is again a contradiction. This concludes the proof of the claim.

After this result, many more followed in the field that is nowadays called Turán-type problems, perhaps the most important being the Erdős-Simonovits-Stone theorem. It determines for all non-bipartite graphs $H$ the asymptotic growth of $\text{ex}(n, H)$ and most surprisingly, it is only determined by its chromatic number $\chi(H)$. Historically, the theorem was divided into two theorems due to Erdős and Simonovits [30] on the one hand, and Erdős and Stone [32] on the other hand. Nowadays, both are bundled into the following theorem.

**Theorem 1.9.** For any graph $H$ we have

$$\text{ex}(n, H) = \frac{1}{2} \left( 1 - \frac{1}{\chi(H) - 1} \right) n^2 + o(n^2).$$

Notice the similarity between this result and that of Turán, using $\chi(K_{r+1}) = r + 1$.

It might be mysterious at first that, when $H$ is not bipartite, the asymptotic order of growth is completely determined by the chromatic number $\chi(H)$ and that the actual shape of $H$ only influences the lower order terms of $\text{ex}(n, H)$. If on the other hand $H$ is bipartite, so $\chi(H) = 2$, we have the following easy corollary.

**Corollary 1.10.** The graph $H$ is bipartite if and only if $\text{ex}(n, H) = o(n^2)$.

This case is commonly referred to as the degenerate case, and we will focus on this case. For a more extensive history, background and results, we again refer the reader to the survey due to Füredi and Simonovits [41].

To prove Theorem 1.9 we will actually prove the following result: for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\frac{1}{2} \left( 1 - \frac{1}{\chi(H) - 1} - \epsilon \right) n^2 \leq \text{ex}(n, H) \leq \frac{1}{2} \left( 1 - \frac{1}{\chi(H) - 1} + \epsilon \right) n^2.$$

This is an equivalent statement, but worded differently.

The upper bound will follow from the following lemma, which contains the brunt of the proof. Its proof however, is rather technical and long, and as such we omit it. In essence, it is due to Erdős and Stone [30].

**Lemma 1.11.** For any $s, t \in \mathbb{N}$ and $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we have

$$\text{ex}(n, T_{st,s}) \leq \frac{1}{2} \left( 1 - \frac{1}{s - 1} + \epsilon \right) n^2.$$
Proof of Theorem 1.9. Now we show how to use the lemma: let $s = \chi(H)$, and take $t$ to be the size of the largest chromatic class of $H$. It follows that for these choices of $s$ and $t$ we have $H \subseteq T_{s,t,s}$ and therefore, $\text{ex}(n, H) \leq \text{ex}(n, T_{s,t,s}) \leq \frac{1}{2} \left(1 - \frac{1}{s-1} + \epsilon \right) n^2$ by Lemma 1.11.

For the lower bound, it suffices to consider the Turán graph $T_{n,r}$ where $r = \chi(H) - 1$. Then this graph is $H$-free and has the required amount of edges as shown in Construction 1.7.

As we saw in the proof, the Turán graphs $T_{n,r}$ give us good lower bounds for $\text{ex}(n, H)$, where $\chi(H) = r + 1$. It turns out that asymptotically, all extremal graphs actually resemble the Turán graphs. Even more, the almost-extremal graphs do so too. This is what the following two theorems say, which we give respectively, as the former determines more or less the structure of the extremal graphs and the latter tells us that graphs which are not extremal, but almost, still resemble the extremal graphs. In both theorems, due to Erdős and Simonovits [26, 66], $r = \chi(H) - 1$.

Theorem 1.12. If $G \in \text{EX}(n, H)$, then $G$ can be obtained from $T_{n,r}$ by adding and deleting $o(n^2)$ edges. Moreover, 
\[ \delta(G) = \left(1 - \frac{1}{r} \right) n + o(n). \]

Notice that when $r$ divides $n$, we have $\delta(T_{n,r}) = n - \frac{n}{r} = \left(1 - \frac{1}{r} \right) n$.

Theorem 1.13. If $G$ is $H$-free then for every $\epsilon > 0$ there exists $\delta > 0$ and $n_0$ such that if $\epsilon(n) > \frac{1}{2} \left(1 - \frac{1}{r} \right) n^2 - \delta n^2$, then $G$ can be obtained from $T_{n,r}$ by adding and deleting at most $\epsilon n^2$ edges whenever $n \geq n_0$.

To conclude, we discuss a result by Simonovits [67] which gives $\text{EX}(n, H)$ asymptotically when $H$ satisfies a certain chromatic property. Suppose $\chi(H) = r + 1$, if $H$ contains an edge $e$ such that $\chi(H - e) = r + 1$, then this edge is called a colour-critical edge. For example, in $K_{r+1}$, every edge is colour-critical. Another example which we will discuss in the next part is the graph $C_{2k+1}$.

Theorem 1.14. If $H$ has a colour-critical edge $e$ and $\chi(H - e) = r \geq 2$, then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $\text{EX}(n, H) = T_{n,r}$.

Actually, in another paper he proves that the converse is also true: if $T_{n,r} \in \text{EX}(n, H)$ for $n$ large enough, then $\chi(H) = r + 1$ and $H$ contains a colour-critical edge [66].

1.2 Some exact Turán numbers

1.2.1 Odd cycles

As already mentioned, there are very few instances where $\text{ex}(n, H)$ is known exactly for all $n$. We encountered already one example when $H = K_{r+1}$. A second one, which we will discuss now, will be the odd cycles $C_{2k+1}$. Because $K_3 = C_3$ is already discussed in Theorem 1.4, we will only consider odd cycles $C_{2k+1}$ where $k \geq 2$. It is clear that every edge of $C_{2k+1}$ is colour-critical, as $C_{2k+1} - e = P_{2k+1}$ for any edge $e$, and hence when $n$ is large enough, we know by Theorem 1.14 that $\text{EX}(n, C_{2k+1}) = T_{n,2}$. Many results on odd cycles were already known but scattered everywhere. Füredi and Gunderson managed to compile these results in one theorem with a rather easy proof and determine the extremal graphs [39].

When $n \leq 2k$, it is immediate that $\text{EX}(n, C_{2k+1}) = K_n$. Moreover, for any $n$, it is clear that $\text{ex}(n, C_{2k+1}) \geq \frac{n^2}{4} = \epsilon(T_{n,2})$. We even know that for $n$ large enough, this is an equality. We

4. By $H - e$ we mean we delete the edge $e$ from $E(H)$. 
will show that ‘large enough’ means \( n \geq 4k \) here.
Let \( n \leq 4k - 1 \), we create a new graph \( F(n, 2) \) on \( n \) vertices by identifying a vertex from each of the two complete graphs \( K_{2k} \) and \( K_{n-2k+1} \) with each other. This graph is indeed \( C_{2k+1} \)-free and has \( (\begin{array}{c} k+1 \\ 2 \end{array}) + \left( n - 2k + 1 \right) \) edges. One can check that this is strictly larger than \( \left\lfloor \frac{n^2}{4} \right\rfloor \) for \( 3 \leq n \leq 4k - 3 \) and equal when \( n = 4k - 2, 4k - 1 \). We can see here already that the value \( n_0 \) from Theorem 1.14 should satisfy \( n_0 > 4k - 1 \) for odd cycles.

Finally, we show another \( C_{2k+1} \)-free graph \( K_k \oplus \overline{K_{n-k}} \) which will turn out to be extremal for certain values of \( n \). Suppose \( n \geq k \), take a complete bipartite graph \( K_{k,n-k} \) and add all edges in the first part, or equivalently as the name suggests, joining a complete graph \( K_k \) and \( K_{n-k} \) (an independent set of size \( n - k \)) by connecting every vertex in the former to every vertex in the latter. One can see that this graph contains no \( C_{2k+1} \) and has \( (\begin{array}{c} k \\ 2 \end{array}) + k(n - k) \) edges. A straightforward computation shows that \( F(n, 2) \) has more edges than \( K_k \oplus \overline{K_{n-k}} \), except when \( n = 3k - 1, 3k \).

Having defined these graphs, we can formulate the theorem.

**Theorem 1.15.** Let \( k \geq 2 \), then

\[
\text{ex}(n, C_{2k+1}) = \begin{cases} 
(\begin{array}{c} n \\ 2 \end{array}) & \text{for } n \leq 2k, \\
(\begin{array}{c} 2k \\ 2 \end{array}) + \left( n - 2k + 1 \right) & \text{for } 2k + 1 \leq n \leq 4k - 1, \\
\left\lfloor \frac{n^2}{4} \right\rfloor & \text{for } 4k - 2 \leq n.
\end{cases}
\]

Moreover, the extremal graphs can be described in the following exhaustive list:

\[
\text{EX}(n, C_{2k+1}) \text{ contains }
\begin{cases} 
K_n & \text{for } n \leq 2k, \\
F(n, 2) & \text{for } 2k + 1 \leq n \leq 4k - 1, \\
K_k \oplus \overline{K_{n-k}} & \text{for } n = 3k - 1, 3k, \\
T_{n,2} = K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor} & \text{for } 4k - 2 \leq n.
\end{cases}
\]

We can conclude that \( n_0 = 4k \) in Theorem 1.14 when \( H = C_{2k+1} \). Note that when \( n = 4k - 2, 4k - 1 \) we have two extremal graphs: \( F(n, 2) \) and \( T_{n,2} \). One can compute that they indeed have an equal amount of edges.

We will prove this result, modulo one lemma which combines a few results on 2-connected \( C_{2k+1} \)-free graphs which might lead us too far.

**Lemma 1.16.** Suppose \( n \geq 2k + 1 \geq 5 \) and \( G \) is a 2-connected, \( C_{2k+1} \)-free, non-bipartite graph with at least \( \left\lfloor \frac{n^2}{4} \right\rfloor \) edges. Then \( e(G) \leq \text{ex}(n, C_{2k+1}) \) and equality holds only if \( n = 3k - 1, 3k \) and \( G = K_k \oplus \overline{K_{n-k}} \).

To prove Theorem 1.15 we decompose \( G \in \text{EX}(n, C_{2k+1}) \) into maximal 2-connected subgraphs, meaning that each subgraph is 2-connected and not properly contained in another 2-connected subgraph. These maximal subgraphs are also called 2-connected blocks. After decomposing the graph into blocks, we rearrange these and apply the lemma to each of these blocks if they satisfy the assumptions. Before we do this, we discuss this decomposition a bit deeper. Recall that if we delete a vertex in a 2-connected graph, we still end up with a connected graph. Now take the extremal \( C_{2k+1} \)-free graph \( G \) from earlier. It is immediate to see that this graph is connected by maximality. If \( G \) is 2-connected, then its 2-connected block decomposition consists only of the block \( G \). If \( G \) is not 2-connected then there exists a cut-vertex \( v \) such that \( G \setminus v \) consists of at least two connected components \( G_1, \ldots, G_m \), which pairwise share the cut-vertex \( v \). Then we can continue inductively on each of these components alone to find the maximal 2-connected subgraphs. In this way, we find blocks \( B_1, \ldots, B_l \) which are all 2-connected and connected by shared cut-vertices. Moreover \( \sum_{i=1}^l (v(B_i) - 1) = v(G) - 1 \).

As a picture says more than a thousand words, the following figure might enlighten what we actually mean by this decomposition. The vertices connecting the blocks are the cut-vertices.
Remark that when \( k = 2 \), the graph becomes \( K_4 \), which is contained in Turán’s result. Therefore, we will assume \( k \geq 3 \) from here on out.

\section{1.2.2 Even wheels}

The next family of graphs we want to discuss are the even wheels. An even wheel \( W_{2k} \) can be constructed from an odd cycle \( C_{2k-1} \), together with one additional vertex which is adjacent to all others. Remark that when \( k = 2 \), the graph becomes \( K_4 \), which is contained in Turán’s result. Therefore, we will assume \( k \geq 3 \) from here on out.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.png}
\caption{A graph and its decomposition into 2-connected blocks \( B_1, \ldots, B_6 \).}
\end{figure}

\textbf{Proof of Theorem 1.15} Suppose \( G \in \text{EX}(n, C_{2k+1}) \), and take its decomposition into 2-connected blocks \( B_1, \ldots, B_l \). Denote \( v(B_i) = n_i \), then \( \sum (n_i - 1) = n - 1 \) and by maximality each block \( B_i \) should have \( \text{ex}(n_i, C_{2k+1}) \geq \left\lceil \frac{n_i^2}{4} \right\rceil \) edges. There are three possibilities for the blocks: either \( B_i \) is a complete graph and \( n_i \leq 2k \), \( B_i \) is bipartite and hence by maximality \( T_{n_i,2} \) or by the lemma we have \( B_i = K_{k} \oplus K_{n_i-k} \) and \( n_i = 3k - 1, 3k \).

If we have only one block, we are done by Lemma 1.16. So assume \( l \geq 2 \), we will have to separate different cases. Again by maximality we have \( n > 2k \).

\textbf{First case.} All blocks are complete graphs, so \( n_i \leq 2k \) for all \( i \). Now consider the graph \( F(n, s) \) on \( n \) vertices which is the natural generalisation of \( F(n, 2) \): the graph consists of \( s \) blocks having one common vertex, all blocks are complete graphs and all but one have \( s \) (size) \( 2k \). It is straightforward to check that \( e(G) \leq e(F(n, s)) \) (rearrange \( G \) such that all its blocks have one vertex in common, this graph has the same number of vertices and edges and makes it easier to compare). Moreover, \( e(F(n, s)) < e(T_{n,2}) \) when \( n > 4k - 1 \) so \( G \) has only two blocks and hence \( G = F(n, 2) \).

\textbf{Second case.} If \( G \) has two blocks \( B_i \) and \( B_j \) that are not both complete subgraphs, then we can create a new \( C_{2k+1} \) graph with more edges than \( G \) and hence \( G \not\in \text{EX}(n, C_{2k+1}) \), which would be a contradiction. We show one subcase explicitly, the rest is completely similar in its computations. Suppose for example that \( B_i \) and \( B_j \) are both complete bipartite graphs and denote their sizes by \( a \) and \( b \) for a less subscript-heavy notation. By maximality they are respectively \( T_{a,2} \) and \( T_{b,2} \) and \( a, b \geq 2k + 1 \). Delete the edges in \( B_i \) and \( B_j \) and replace them by a copy of \( T_{a+b-1,2} \) in \( B_i \cup B_j \) (recall that \( B_i \) and \( B_j \) share a cut-vertex). By

\[
e(T_{a,2}) + e(T_{b,2}) = \left\lceil \frac{a^2}{4} \right\rceil + \left\lceil \frac{b^2}{4} \right\rceil < \left\lceil \frac{(a+b-1)^2}{4} \right\rceil = e(T_{a+b-1,2}),
\]

we find our contradiction. In all other possibilities except when \( B_i = K_k \oplus K_{a-k} \) and \( B_j = K_b \) with \( 2 \leq b \leq k \), we remove the edges in \( B_i \) and \( B_j \) and put \( T_{a+b-1,2} \) in \( B_i \cup B_j \). In this last possibility, we replace them by \( K_{2k} \) and \( K_{a+b-2k} \). One can compute that we always find a graph with more edges than the original one. This completes the second case and hence the proof. \hfill \Box

The next family of graphs we want to discuss are the even wheels. An even wheel \( W_{2k} \) can be constructed from an odd cycle \( C_{2k-1} \), together with one additional vertex which is adjacent to all others. Remark that when \( k = 2 \), the graph becomes \( K_4 \), which is contained in Turán’s result. Therefore, we will assume \( k \geq 3 \) from here on out.
As we can see in the above figure, the graphs $W_{2k}$ have a colour-critical edge and hence by Theorem 1.14 for $n$ large enough we have $EX(n, W_{2k}) = T_{n,3}$. This has partially been proved by Dzido [24]. In order to do this, he makes a reduction to $C_{2k-1}$ and uses this to show the following theorem.

**Theorem 1.17.** For all $k \geq 3$ and $n \geq 6k - 10$,

$$\text{ex}(n, W_{2k}) = \left\lfloor \frac{n^2}{3} \right\rfloor.$$

It is important to note that Dzido’s paper came before the one on odd cycles by Füredi and Gunderson [39] from the previous section. Therefore, using this new, compacter result, one can try to extend Dzido’s result. Unfortunately, at the time of writing, we have not been able to do so.

To be fully clear, in this section we use the notation $C_{2k-1}$ instead of $C_{2k+1}$, therefore we need to slightly adjust the result from the previous section:

$$\text{ex}(n, C_{2k-1}) = \left\lfloor \frac{n^2}{4} \right\rfloor$$

when $n \geq 4k - 6$.

**Proof of Theorem 1.17** The lower bound can immediately be given by $T_{n,3}$, so it remains to prove the upper bound.

Let $G$ be a graph on $n \geq 6k - 10$ vertices and $\left\lfloor \frac{n^2}{3} \right\rfloor + 1$ edges. Suppose it is $W_{2k}$-free, we will try to find a contradiction.

It is clear that $\Delta(G) \geq n - \left\lfloor n/3 \right\rfloor$ as otherwise

$$e(G) \leq \frac{1}{2}n(n - \left\lfloor n/3 \right\rfloor - 1) < \left\lfloor \frac{n^2}{3} \right\rfloor + 1 = e(G).$$

Take a vertex $v$ with $d(v) = \Delta(G) = n - p$, where, by the above, $1 \leq p \leq \left\lfloor n/3 \right\rfloor$. Now we consider the subgraph $H$ induced by the vertices of $N(v)$. This subgraph has $d(v) = n - p \geq 4k - 6$ vertices and so cannot contain $C_{2k-1}$, as this, together with $v$ would produce $W_{2k}$. Therefore, by the formula we gave above

$$e(H) \leq \left\lfloor \frac{(n - p)^2}{4} \right\rfloor.$$
Using this, we have an estimate for the number of edges of $G$:

$$\left\lfloor \frac{n^2}{3} \right\rfloor + 1 = e(G) \leq e(H) + p(n-p) = \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + p(n-p),$$

as we obtain $H$ from $G$ by deleting $p$ vertices which have degree at most $n-p$. A straightforward calculation shows that the right hand side achieves it maximum when $p = \lfloor n/3 \rfloor$ and this maximum is $\lfloor n^2/3 \rfloor$, a contradiction. \hfill $\square$

Using this result, Dzido considered the case $k = 3$ and managed to completely determine this case.

**Proposition 1.18.** For all $n \geq 6$,

$$\text{ex}(n, W_6) = \left\lfloor \frac{n^2}{3} \right\rfloor.$$

The proof is a simple corollary of the previous theorem and a computation by hand for $n = 6, 7$, we will not replicate it here.

Seeing this result, one could naively hope that the main result for even wheels also holds for any $n \geq 2k$. However, a closer look at the proof tells us that this might be a bridge too far. It relies on the determination of $\text{ex}(n, C_{2k-1})$, and this is split up in three cases as we saw in the previous section. It is hence to be expected that this result cannot be readily generalised for $n \geq 2k$, as we can also see in the next proposition.

**Proposition 1.19.** For all $k \geq 3$,

$$\text{ex}(2k, W_{2k}) = 2k^2 - 2k.$$

**Proof.** Any $(2k-2)$-regular graph on $2k$ vertices reaches this bound and is clearly $W_{2k}$-free as no vertex has degree $2k-1$. For example, consider the complete graph $K_{2k}$ and remove $k$ independent edges (edges having no common vertex). This graph indeed has

$$\frac{2k}{2} - k = 2k^2 - 2k$$

edges and is $W_{2k}$-free as no vertex has degree $2k-1$.

Now assume we have a $W_{2k}$-free graph $G$ on $2k$ vertices and $2k^2 - 2k + 1$ edges. Similarly as in the previous proof, we can estimate the maximum degree $\Delta(G)$: it has to be $2k-1$ as

$$\frac{1}{2}(2k)(2k-2) < 2k^2 - 2k + 1.$$  

So take $v$ with $d(v) = 2k-1$ and the induced subgraph $H$ on $N(v)$ (which is $G$ with $v$ deleted). This graph has $2k-1$ vertices and $2k^2 - 4k + 2$ edges. We know from Theorem 1.15 that for $k \geq 3$,

$$\text{ex}(2k-1, C_{2k-1}) = \left( \frac{2k-2}{2} \right) + 1 = 2k^2 - 5k + 5.$$  

This is smaller than $e(H)$ when $k \geq 3$ so $H$ contains $C_{2k-1}$ and hence $G$ contains $W_{2k}$.

\hfill $\square$

Actually, we can prove that the $W_{2k}$-free graph on $2k$ vertices we constructed above is the unique extremal graph, which also shows that there is only one $(2k-2)$-regular graph on $2k$ (this itself is an easy exercise). The reasoning consists of two easy steps: first show that there cannot be a vertex $v$ with $d(v) \leq 2k-3$, using $e(G) \geq k\delta(G)$. Then compute the number of vertices with degree $2k-1$ and the number with degree $2k-2$ and find that all of them should have degree $2k-2$.

Comparing the results we found for $\text{ex}(2k, W_{2k})$, we see that this last bound is bigger than $\lfloor (2k)^2/3 \rfloor$ when $k \geq 4$. For $k = 3$ we find that our construction coincides with $T_6,3$. The question for which $n$ it holds that $\text{ex}(n, W_{2k}) = \lfloor n^2/3 \rfloor$ still remains open.

**Question 1.20.** Determine $\text{ex}(n, W_{2k})$ and $\text{EX}(n, W_{2k})$ when $k \geq 3$ and $2k + 1 \leq n \leq 6k - 11$.  

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1.2.3 Intersecting triangles

The last non-bipartite case we will consider is the case when the forbidden graph consists of \( k \) triangles intersecting in one vertex. This graph is denoted by \( F_{2k+1} \) and sometimes called a \( k \)-fan. In contrast to the previous two examples, this graph has no colour critical edges so we cannot apply Simonovits’ colour-critical edge theorem. On the other hand, its chromatic number is 3, so by the Erdős-Stone-Simonovits theorem we know that the main term in \( \text{ex}(n, F_{2k+1}) \) should be \( \lfloor n^2/4 \rfloor \). Füredi, Erdős, Gould and Gunderson \[27\] managed to describe the behaviour of \( \text{ex}(n, F_{2k+1}) \) precisely, when \( n \) is large enough.

**Theorem 1.21.** Let \( n \geq 50k^2 \), then

\[
\text{ex}(n, F_{2k+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} k^2 - k & \text{for } k \text{ odd}, \\ k^2 - \frac{3}{2}k & \text{for } k \text{ even}. \end{cases}
\]

Remark that this is indeed a generalisation of Mantel’s result, i.e. when \( k = 1 \). We will not prove the upper bound as this is a rather long proof, but we will show the lower bound. The lower bound is given by the following graphs. If \( k \) is odd and \( n \geq 4k - 1 \), we take \( T_{n,2} \) and add the edges of two vertex-disjoint copies of \( K_k \) in one side. If \( k \) is even and \( n \geq 4k - 3 \) we take \( T_{n,2} \) and add the edges of a graph on \( 2k - 1 \) vertices, \( k^2 - (3/2)k \) edges and \( \Delta(G) = k - 1 \) (obtained for example by adding \( k - 1 \) indepent edges to a \( (k - 2) \)-regular graph on \( 2k - 1 \) vertices) in one side. Moreover, they claim to be able to show that these are the only maximal \( F_{2k+1} \)-free graphs when \( n \geq 50k^2 \). Finally, they conjecture that the result also holds true when \( n \geq 4k \). We will explain why they propose \( n \geq 4k \) and not \( n \geq 2k + 1 \). For the proof, we need a result by Erdős and Gallai which is interesting in its own, but which we will only mention here. We have already used the concept of \( k \) independent edges, i.e., \( k \) edges having pairwise no vertex in common. This is also called a matching of size \( k \) (as \( k \) vertices are ‘matched’ with \( k \) others) and we denote this graph by \( M_{2k} \).

**Theorem 1.22.** For all \( k \geq 1 \),

\[
\text{ex}(n, M_{2k}) = \max \left\{ \binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1) \right\}.
\]

Using this result, we can make a reduction similar as we did in the last section about odd wheels, as \( F_{2k+1} \) is nothing more than a matching \( M_{2k} \) with an added vertex, which is adjacent to every other.

**Proposition 1.23.** For all \( k \geq 1 \),

\[
\text{ex}(2k+1, F_{2k+1}) = 2k^2 - 1.
\]

**Proof.** Take the complete graph on \( 2k + 1 \) vertices and delete \( k + 1 \) edges: \( k \) independent edges and one more from the vertex not covered by the other \( k \) edges. Since the maximum degree is \( 2k - 1 \), the graph is \( F_{2k+1} \)-free and has \( \binom{2k+1}{2} - (k+1) = 2k^2 - 1 \) edges.

Now take a graph \( G \) on \( 2k + 1 \) vertices with \( 2k^2 \) edges and assume it is \( F_{2k+1} \)-free. Like we have done already a few times now, one can compute that \( \Delta(G) \geq 2k \). Now take a vertex \( v \) of maximum degree and consider the subgraph \( H \subseteq G \) induced by the vertices of \( N(v) \),

\[
e(H) = 2k^2 - 2k > \max \left\{ \binom{2k-1}{2}, \binom{k-1}{2} + 2k^2 - 1 \right\} = \text{ex}(2k, M_{2k}).
\]

It follows that \( H \) contains a matching of size \( k \) and therefore \( G \) contains \( F_{2k+1} \), which gives us the contradiction we wanted and concludes the proof. \( \square \)
When $k$ is odd, this is larger than the bound stated in Theorem 1.21 and therefore the conjecture would already be false if one uses $n \geq 2k + 1$ instead of $n \geq 4k$.

In [27] the smallest case $k = 2$ has also been fully determined. There the restriction $n \geq 50k^2 = 200$ can be reduced to $n \geq 5$. Obviously when $n \leq 4$ the extremal graph is the complete graph $K_n$. A lot of cases remain open.

**Question 1.24.** Determine $\text{ex}(n, F_{2k+1})$ for all $k \geq 3$ and $2k + 1 \leq n \leq 50k^2$. Determine $\text{EX}(n, F_{2k+1})$ for all $k \geq 2$ and $2k + 1 \leq n \leq 50k^2$.

### 1.2.4 Trees

Leaving behind non-bipartite graphs, we solely focus on the bipartite case from here on out. One of the easiest classes of bipartite graphs is the class of trees. Recall that this is the class of connected graphs containing no cycles. An easy first result is the following. Denote by $T_k$ an arbitrary tree on $k$ vertices.

**Proposition 1.25.** For all $k \geq 2$,

$$\text{ex}(n, T_k) < (k - 2)n$$

In the following proof we use the method of embedding one graph $H$ into another graph $G$. By this we mean that we construct an injective map $\phi$ from $V(H)$ to $V(G)$ such that two adjacent vertices in $H$ get mapped to adjacent vertices of $G$. This means that we find $H$ as a subgraph of $G$. If $v \in V(H)$, we denote its embedding by $\phi(v)$. We use this convention only here.

**Proof.** First we prove that every graph $G$ with minimum degree at least $k - 1$ contains every tree $T_k$ on $k$ vertices. We do this by greedily embedding the tree in $G$. Take a vertex $v$ of $T_k$ and embed it anywhere in $G$. Next, take any vertex $w$ adjacent to $v$ and embed it in $G$. This means we map $w$ to any vertex adjacent to $\phi(v)$. Continue in this way, taking vertices adjacent to already embedded ones and embed them. Suppose we have already embedded $i < k$ vertices. Take a vertex $r$ in $T_k$ adjacent to $s$, an already embedded one. We have to identify $r$ with a vertex adjacent to $\phi(s)$. Since the minimum degree of $G$ is at least $k - 1$, the degree of $\phi(s)$ is at least $k - 1$. At most $i - 1$ of its $k - 1$ neighbors are already taken by the other embedded vertices. This means that there are $k - 1 - (i - 1) \geq 1$ vertices available, so we can embed $r$ to one of these.

Now we are ready to prove the theorem. We do this by induction on $n$. For $n = k$ this is obvious. Take a graph $G \in \text{EX}(n, T_k)$. Removing a vertex with minimum degree $\delta(G)$ we find the following inequalities.

$$\text{ex}(n, T_k) - (k - 2) \leq \text{ex}(n, T_k) - \delta(G) \leq \text{ex}(n - 1, T_k) < (k - 2)(n - 1).$$

The first inequality is because the graph has minimum degree at most $k - 2$ by the discussion above. The second is because $G$ didn’t contain $T_k$ as a subgraph and therefore $G$ with a vertex deleted also does not contain it. The last one is by induction. This concludes the proof. 

Among all trees, two classes of trees stand out as the easiest to deal with: stars and paths. We will discuss both of them. Recall that the star $S_k$ is isomorphic to $K_{1,k-1}$. We will return to this fact in Chapter 3. If $G$ does not contain $S_k$ as a subgraph, it means we are bounding the maximum degree of $G$. Therefore, following theorem is quite straightforward.

**Proposition 1.26.** For all $k \geq 2$,

$$\text{ex}(n, S_k) = \left\lfloor \frac{1}{2} (k - 2)n \right\rfloor.$$
Remark that when \( n \leq k - 1 \), the extremal graph is \( K_n \) having \( \frac{1}{2} n(n - 1) \) edges, which is at most \( \frac{1}{2} n(k - 2) \). Therefore, we do not need to differentiate between the trivial case \( n \leq k - 1 \) and the case \( n \geq k \) as we had to do in the previous sections.

**Proof.** For the lower bound, we can assume that \( n \geq k \) as otherwise the bound is trivial. If either \( k \) or \( n \) are even, we can construct the \((k - 2)\)-regular graph on \( n \) vertices. This graph does not contain \( S_k \) and has the required amount of edges. If both \( k \) and \( n \) are odd, we can construct the \((k - 2)\)-regular graph on \( n + 1 \) vertices. Deleting a vertex from this graph removes \( k - 2 \) edges. Consider the \( k - 2 \) vertices which were adjacent to the deleted vertex. We can add edges back by adding independent edges between \( k - 1 \) of them (as \( k - 1 \) is even). Clearly, the maximum degree of the constructed graph \( G \) is again at most \( k - 2 \) and hence does not contain \( S_k \). We count the edges by counting the degrees:

\[
2e(G) = (n - 1)(k - 2) + k - 3,
\]

from which the lower bound follows.

To prove the upper bound, let \( G \) be a graph on \( n \) vertices \( \{v_1, \ldots, v_n\} \) containing no \( S_k \). As we said before, the degree of a vertex \( v_i \) is bounded from above by \( k - 2 \), as a vertex with degree at least \( k - 1 \) gives rise to a subgraph \( S_k \). Hence

\[
2e(G) = \sum_{i=1}^{n} d(v_i) \leq n(k - 2).
\]

\( \qed \)

For the case of the path \( P_k \), Erdős and Gallai [28] derived in 1959 the same bound.

**Proposition 1.27.** For all \( k \geq 2 \),

\[
ex(n, P_k) \leq \frac{1}{2}(k - 2)n.
\]

The original proof by Erdős and Gallai is not complicated, but rather long, relying on several intermediate lemmas. We will give a different proof.

**Proof.** Consider a graph \( G \) on \( n \) vertices having more than \( \frac{1}{2}(k - 2)n \) edges. If \( G \) contains a vertex with degree smaller than \( \frac{1}{2}(k - 2) \), we can remove it and find a graph on \( n - 1 \) vertices and having more than \( \frac{1}{2}(k - 2)(n - 1) \) edges. Continuing in this way, we find a subgraph \( H \) on \( n' \) vertices having more than \( \frac{1}{2}(k - 2)n' \) edges and minimum degree at least \( \frac{1}{2}(k - 2) \). This subgraph is non-empty: suppose that we would have removed all vertices in this way, then this means that every vertex has degree at most \( \frac{1}{2}(k - 2) \), but then \( G \) could have at most \( \frac{1}{2}(k - 2)n \) edges, which is in contradiction with our assumption. Moreover, we can assume that \( H \) is connected, otherwise we can restrict ourselves to the connected component with the largest ratio of edges to vertices and call this \( H \). Finally, in order for \( H \) to have more than \( \frac{1}{2}(k - 2)n' \) edges, we need \( k - 1 < n' \).

We will show that we can find a path on \( k \) vertices in \( H \). Take a path of maximum length in \( H \), say \( v_1 \ldots v_l \). If \( l \geq k \), then we are done, so suppose \( l \leq k - 1 \). By maximality, all the neighbours of \( v_1 \) and \( v_l \) are in the path. We divide in two cases depending on whether \( v_1 \) is adjacent to \( v_l \) or not.

1. **First case.** Assume \( v_1 \sim v_l \), then \( v_1 \ldots v_l v_1 \ldots v_{l-1} \) is also a path of maximum length and thus all neighbours of \( v_i \) are in the path, for any \( i \). Then, because \( H \) is connected, \( V(H) = \{v_1, \ldots, v_l\} \), \( k - 1 < n' = l \leq k - 1 \), a contradiction.

2. **Second case.** So suppose \( v_1 \not\sim v_l \), then, because the minimum degree is \( \frac{1}{2}(k - 2) \), there exists an \( i \) such that \( v_i \sim v_{i+1} \) and \( v_i \sim v_l \) by the pigeonhole principle. Hence \( v_1 \ldots v_i v_{i+1} \ldots v_{l-1} v_1 \) is a cycle in \( H \), and reasoning as before we find that \( V(H) = \{v_1, \ldots, v_l\} \), giving us the same contradiction. \( \qed \)
If $k - 1$ divides $n$, then the bound can be achieved by $\frac{n}{k - 1}$ disjoint copies of $K_{k-1}$.

This theorem has been extended by Faudree and Schelp [33], and independently by Kopylov [48] to the case when $n$ is not divisible by $k - 1$.

**Theorem 1.28.** Let $n \equiv r \pmod{k - 1}$, $0 \leq r < k - 1$. Then for all $k \geq 2$,

$$\text{ex}(n, P_k) = \frac{1}{2}(k - 2)n - \frac{1}{2}r(k - 1 - r).$$

We will not prove this result, we refer the reader to the papers mentioned above.

Looking back at Theorems 1.26 and 1.27 it is tempting to hope that the bound might hold for any tree on $k$ vertices, and not only $S_k$ or $P_k$. This is exactly the content of the following conjecture made by Erdős and Sós in 1963.

**Conjecture 1.29.** For all $k \geq 2$,

$$\text{ex}(n, T_k) \leq \frac{1}{2}(k - 2)n.$$

This important conjecture has been proved partially by Ajtai, Komlós, Simonovits and Szemerédi in a series of three papers [4], which are yet to be published. More precisely, their result is the following.

**Theorem 1.30.** There exists an integer $k_0$ such that if $k > k_0$ then

$$\text{ex}(n, T_k) \leq \frac{1}{2}(k - 2)n.$$

**Question 1.31.** Find an explicit value for $k_0$ in Theorem 1.30. Is the conjecture still true if $3 \leq k \leq k_0$?
Chapter 2

The quadrilateral

As we saw in Chapter 1, Theorem 1.9 determines the asymptotic growth of $\text{ex}(n, H)$ for all non-bipartite graphs $H$. Therefore, when studying Turán numbers it is natural to take a closer look at the smallest bipartite graphs (remark that we only consider connected graphs). We already took care of trees in the last chapter, so the next case to consider would be $H = C_4 = K_{2,2}$, the quadrilateral. Before we dive into the results, we need some introduction on projective geometry and in particular, projective planes. We assume that the reader has previous knowledge on finite fields and vector spaces and is able to work with them. Preferably, he has also come across projective geometries before.

2.0 Foundations: projective geometry

Unfortunately, unlike the previous chapter and the chapters to come, we cannot provide a self-contained introduction to this chapter. Projective geometry is just a far too large subject to do so. For a more complete introduction we refer the reader to the standard references by Hirschfeld [42, 43].

2.0.1 Projective planes and projective spaces

First we recall some properties of finite fields. Let $q = p^h$, $p$ prime and $h \geq 1$, then the finite field with $q$ elements will be denoted as $GF(q)$. One can define the trace function $Tr(x)$ and the norm function $N(x)$ in the following way:

$$ Tr(.) : GF(q) \to GF(p) : Tr(x) = \sum_{i=0}^{h-1} x^{p^i}, $$

$$ N(.) : GF(q) \to GF(p) : N(x) = \prod_{i=0}^{h-1} x^{p^i}. $$

It is immediate that for $x, y \in GF(q)$, $Tr(x + y) = Tr(x) + Tr(y)$ and $N(xy) = N(x)N(y)$. Both of these functions will be used in constructions of graphs or certain sets of points in projective planes. One property related to the trace function is the following standard result in finite field theory.

**Proposition 2.1.** If $q$ is even, then the equation

$$ x^2 + x + a = 0, $$

$a \in GF(q)$ has two solutions in $GF(q)$ if and only if $Tr(a) = 0$, otherwise it has zero solutions in $GF(q)$. 


For any finite field $GF(q)$, we can define the \textit{n-dimensional vector space over $GF(q)$}, denoted by $V(n, q)$. Recall that $GF(q)$ itself can be identified with $V(h, p)$. If $x \in V(n, q)$ then it has coordinates $x = (x_0, \ldots, x_{n-1})$, where $x_i \in GF(q), 0 \leq i \leq n - 1$. Note that we start counting from $x_0$ and not from $x_1$.

Starting from vector spaces over finite fields, we could already define projective planes and projective spaces. However we prefer to do so and take a more general route, at least in the case of projective planes. We will define projective planes axiomatically and construct a certain class using these vector spaces. For projective spaces, it matters very little to us which approach we take, as we will explain later on. A projective plane is a special structure of points and lines; this is our starting point.

A point-line geometry is a triple $(P, L, I)$: a non-empty set of points $P$, a set of lines $L$ (disjoint from $P$) and an incidence relation $I \subseteq P \times L$ such that for every $L \in L$ there are at least two $x \in P$ such that $(x, L) \in I$. If $(x, L) \in I$, we also write $x \in I \subseteq P$ and say that $x$ is incident with $L$, $x$ lies on $L$, $L$ is incident with $x$, $L$ goes through $x$ or $L$ contains $x$. We can define the incidence graph of a point-line geometry in the following way: it is a bipartite graph with classes $P$ and $L$ and $x \sim L$ if and only if $x \in I \subseteq L$. The incidence matrix of a point-line geometry is a matrix $M$ where the rows and columns are indexed by the points $x_1, \ldots, x_{|P|}$ and the lines $L_1, \ldots, L_{|L|}$, respectively, and the entries $m_{ij}$ are given by

$$m_{ij} = \begin{cases} 0 & \text{if } (x_i, L_j) \notin I \\ 1 & \text{if } (x_i, L_j) \in I. \end{cases}$$

Two points are \textit{collinear} if there exists a line containing them both. A subspace is a set of points such that if two points of a line belong to the set, then every point of the line belongs to the set. Clearly $P$ is a subspace. The \textit{subspace generated by a set of points $S$} is the smallest subspace containing this set and denoted by $\langle S \rangle$. A \textit{maximal subspace} is a subspace that is not properly contained in another subspace.

A point-line geometry is a partial linear space if every two points are incident with at most one line. It is a linear space if every two points are incident with exactly one line. In a partial linear space we can identify the lines with the set of points they contain. In this way the incidence relation becomes the containment relation. Therefore, we might write $x \in L$ when we have $x \in I \subseteq L$. Moreover, if $x$ and $y$ are collinear points, the line containing them both is unique and we can denote it as $xy$.

These definitions allow us to define projective planes in a compact way.

\textbf{Definition 2.2.} A projective plane is a linear space such that

\begin{enumerate}[(PP2)]
\item any two distinct lines meet in a unique point,
\item there exist four points of which no three are collinear.
\end{enumerate}

From these axioms, it can be shown that there exists a value $n \geq 2$ such that every line contains $n + 1$ points and every point is incident with $n + 1$ lines. This $n$ is called the order of the plane. If a projective plane has order $n$, we can count its number of points and lines. Take a point $x$ and consider the $n + 1$ lines through $x$. Each of these lines has $n$ points, different from $x$. As each point of the plane is on exactly one of these lines, it follows that the number of points is $(n + 1)n + 1 = n^2 + n + 1$. As an exercise, one can start from a line and deduce that the total number of lines is also $n^2 + n + 1$.

We see that the subspaces of a projective plane are the empty set, a point, a line and the whole projective plane. Their dimensions are respectively $-1, 0, 1$ and $2$, where the \textit{dimension of a subspace $X$}, denoted as $\dim(X)$, is the largest value $n$ such that there exists a strictly ascending chain of subspaces of this form: $\emptyset = X_{-1} \subsetneq X_0 \subsetneq \cdots \subsetneq X_n = X$. This coincides with the intuition that a plane is 2-dimensional and lines are really maximal subspaces of a plane.
We are now ready to give the promised construction of a projective plane, starting from a vector space over a finite field.

**Construction 2.3.** For any prime power $q$, consider the vector space $V(3, q)$. Then we define a projective plane in the following way: the points are the 1-dimensional subspaces of $V(3, q)$, the lines are the 2-dimensional subspaces of $V(3, q)$, and incidence is containment. One can check that this indeed defines a projective plane of order $q$, which will be denoted as $PG(2, q)$.

Remark that the notation $PG(2, q)$ implies that $q$ is a prime power. The projective plane $PG(2, q)$ is often called the Desarguesian projective plane. This implies that there also exist non-Desarguesian projective planes, but we will not discuss those here. Whenever we talk about a projective plane, we will mean the (finite) Desarguesian projective plane from now on. It is interesting that also the known finite non-Desarguesian planes all have prime power orders. One of the longest-standing conjectures in finite geometry states that if the order of a finite projective plane is always a prime power. At this moment, we are far away from a proof. Tarry proved that there exists no projective plane of order 6 in 1901 when answering Euler’s famous thirty-six officers problem in the negative. Only 90 years later, Lam, Thiel and Swiercz [50] proved that there is no plane of order 10. As of now, it is unknown whether there exists a projective plane of order 12. The only criterion for non-existence of projective planes is given by the Bruck-Ryser theorem [15].

**Theorem 2.4.** If a projective plane of order $n$ exists, where $n \equiv 1, 2 \mod 4$, then $n$ is the sum of two squares.

This implies for example that a projective plane of order 14 cannot exist. This theorem does not exclude the possibility of a projective plane of order 10, so it’s clearly not a sufficient theorem.

We return to $PG(2, q)$. In $V(3, q)$, we have coordinates to our disposal. We can try to transfer these to $PG(2, q)$. As the 1-dimensional subspaces of $V(3, q)$ are of the form $\{(\lambda x_0, \lambda x_1, \lambda x_2) \mid \lambda \in GF(q)\}$, for certain $(x_0, x_1, x_2) \neq (0, 0, 0)$, we can identify the corresponding projective point with the homogeneous coordinates $(x_0, x_1, x_2)$, up to non-zero multiplicative constant. This means that $(x_0, x_1, x_2)$ and $(\lambda x_0, \lambda x_1, \lambda x_2)$, $\lambda$ non-zero, denote the same point. Moreover $(0, 0, 0)$ is never a point in $PG(2, q)$.

Lines in $PG(2, q)$ can be defined by homogeneous linear equations in $(X_0, X_1, X_2)$. For example, the line $X_2 = 0$ corresponds to the 2-dimensional subspace $\{(a, b, 0) \mid a, b \in GF(q)\}$. In general, a line is defined by an equation $x_0 X_0 + x_1 X_1 + x_2 X_2 = 0$, $(x_0, x_1, x_2) \neq (0, 0, 0)$, and the line is coordinatised as $[x_0, x_1, x_2]$.

Finally, whenever $q$ is a square, we can find a subplane of order $\sqrt{q}$ inside $PG(2, q)$. For example, using all the points with coordinates in $GF(\sqrt{q})$ (up to a scalar), and the lines containing at least two of them, we can construct a projective plane of order $\sqrt{q}$. Note that we do not use all the points of a line. This is an instance of the practical use of an abstract point-line geometry, where lines are objects on their own, rather than subsets of points. This subplane of order $\sqrt{q}$ is called a Baer subplane.

After this discussion on projective planes, we could do the same for projective spaces, but we will not. Instead we give the construction of a Desarguesian projective space $PG(n, q)$, $n \geq 2$.

**Construction 2.5.** For any prime power $q$, consider the vector space $V(n + 1, q)$. The points and lines of $PG(n, q)$ are the 1-dimensional and 2-dimensional subspaces of $V(n + 1, q)$ respectively. In general, the $d$-dimensional subspaces are the $(d + 1)$-dimensional subspaces of $V(n + 1, q)$, where $0 \leq d \leq n$. Incidence is again containment.

A 2-dimensional and $(n - 1)$-dimensional subspace of $PG(n, q)$ are called a plane and hyperplane respectively. Clearly for $n = 2$ this is the same as in Construction 2.3. When $n \geq 3$ one could ask if there are also non-Desarguesian projective spaces. This is the content of the Veblen-Young theorem. To state this theorem, we need two more definitions. First of all, the dimension of a projective space is $\dim(M) + 1$, where $M$ is a maximal non-trivial subspace. For example, the dimension of $PG(n, q)$ is...
n. Secondly, two point-line geometries \((\mathcal{P}, \mathcal{L}, I)\) and \((\mathcal{P}', \mathcal{L}', I')\) are isomorphic if there exists a pair \((\phi_1, \phi_2)\) such that \(\phi_1 : \mathcal{P} \to \mathcal{P}'\) and \(\phi_2 : \mathcal{L} \to \mathcal{L}'\) are bijections and \(x I L\) if and only if \(\phi(x) I' \phi(L)\).

**Theorem 2.6.** Any finite projective space of dimension \(n \geq 3\) is isomorphic to \(PG(n, q)\).

Therefore, the only possible finite projective spaces are the Desarguesian ones, at least when the dimension is at least 3. We note that whenever we write about a projective plane or projective space, we mean \(PG(n, q)\), \(n \geq 2\). In an informal way, we obtained these by the projection of a vector space from its origin.

### 2.0.2 Polarities of projective spaces

As is the case with many symmetric objects, projective spaces have also been studied through their automorphisms. Although we can define most concepts for projective spaces \(PG\), as is the case with many symmetric objects, projective spaces have also been studied through their polarities.

**Definition 2.7.** A collineation of a projective plane is a bijection of the plane to itself, mapping points to points, lines to lines, and preserving incidence.

This means that if \(\alpha\) is a collineation, then, from \(x \in L\), we get \(x^\alpha \in L^\alpha\). If we compose two collineations, we again obtain a collineation. One can check the other axioms to see that the set of collineations form a group, which we denote by \(PGL(3, q)\). We show two examples of collineations.

First of all, a linear transformation of the underlying \(V(3, q)\) gives rise to a collineation of \(PG(2, q)\). If \(M\) is the matrix corresponding to the linear transformation, then \(\mu : \mathcal{P} \to \mathcal{P} : x \mapsto y, y^T = Mx^T\) is the corresponding collineation, where we use the homogeneous coordinates of \(x^T = (x_0, x_1, x_2)^T\). If \(M = \lambda M', \) where \(\lambda \in GF(q) \setminus \{0\}\), then \(M\) and \(M'\) give rise to the same collineation. This set of collineations forms a subgroup of \(PGL(3, q)\), called the projective linear group, and is denoted by \(PGL(3, q)\). Each such collineation is represented by an invertible \(3 \times 3\) matrix (again, determined up to a non-zero scalar). Another example is related to the field automorphisms of \(GF(q)\). If \(\sigma \in \text{Aut}(GF(q))\) then applying \(\sigma\) to any point \(x = (x_0, x_1, x_2)\) by \(x^\sigma = (x_0^\sigma, x_1^\sigma, x_2^\sigma)\) also gives rise to a collineation.

The importance of these two examples is shown by the fundamental theorem of projective geometry.

**Theorem 2.8.** If \(\alpha \in PGL(3, q)\), then \(\alpha = \mu \circ \sigma\), where \(\mu \in PGL(3, q)\) and \(\sigma \in \text{Aut}(GF(q))\).

In this thesis we are mostly interested in the projective linear group and not so much in the field automorphisms, therefore we will rarely use the full group \(PGL(3, q)\).

The dual of a projective plane \(P\) is the projective plane obtained by switching the roles of points and lines of \(P\). One can check that the axioms of a projective plane are indeed satisfied. When \(P = PG(2, q)\), the projective plane is isomorphic to its dual. This implies that when we prove a property of the points of \(PG(2, q)\), we can translate it to a property of the lines of \(PG(2, q)\), using this duality.

**Definition 2.9.** A correlation of a projective plane is a bijection from the plane to itself, mapping points to lines, lines to points, and preserving incidence.

In this case, it means that if \(\beta\) is a correlation, then from \(x \in L\) we get \(L^\beta \in x^\beta\). For example, the map given in coordinates by \((x_0, x_1, x_2) \leftrightarrow [x_0, x_1, x_2]\) is a correlation. This map has the additional property that applying it twice, we get the identity. Correlations possessing this property will be one of the main objects in this thesis.

**Definition 2.10.** A polarity of \(PG(2, q)\) is a correlation of order two.
We can also define polarities for projective spaces $\mathbb{P}G(n,q)$. There, a correlation is defined as a permutation reversing incidence of the subspaces of $\mathbb{P}G(n,q)$. In particular, $\emptyset$ is mapped to $\mathbb{P}G(n,q)$, a point to a hyperplane, a line to an $(n-2)$-space, etc. A polarity remains the special case when the correlation has order two.

Let $\beta$ be a polarity of $\mathbb{P}G(2,q)$, then the image $x^\beta$ of a point $x$ will be commonly denoted as $x^\perp$ and is sometimes called its polar image. Similarly, we can define $L^\perp$ for a line $L$ and see that $x \in L$ if and only if $x^\perp \in L^\perp$. For polarities of projective spaces, we can define $S^\perp$ for subspaces $S$ in the same way.

In terms of a well-chosen incidence matrix of $\mathbb{P}G(2,q)$, a polarity is given by a symmetric incidence matrix, where the point $x_i$ is mapped to the line $L_i$.

Let $\beta$ be a polarity of $\mathbb{P}G(2,q)$. A point $x$ is absolute if $x \in x^\perp$. Similarly a line $L$ is absolute if $L^\perp \in L$. If $x$ is an absolute point, then $x^\perp$ is an absolute line and vice versa. A polarity of $\mathbb{P}G(2,q)$ always has absolute points, as shown by Baer [7].

**Theorem 2.11.** Let $P$ be a projective plane of order $n$, and $\beta$ a polarity of $P$. Then the following holds

- Every absolute line contains exactly one absolute point.
- There are at least $n+1$ absolute points.

This means that if we look at a polarity in matrix terms, we see that there are at least $n+1$ ones on the diagonal.

Moreover, if there are exactly $n+1$ absolute points, he has given an explicit description of them.

**Theorem 2.12.** Let $P$ be a projective plane of order $n$, and $\beta$ a polarity of $P$ with $n+1$ absolute points. Then the following holds

- $n$ is even if and only if all the absolute points all on a line,
- $n$ is odd if and only if no line contains more than two absolute points.

We will look more in depth at polarities and construct graphs related to them in this chapter. Therefore, in order to be able to progress swiftly, we will lay most of the groundwork here.

We will only consider polarities of $\mathbb{P}G(2,q)$ with $q+1$ points. One can show that in this case, the polarity corresponds to a non-degenerate symmetric bilinear form of $V(3,q)$. This means that we have an invertible matrix to our disposal, defining the polarity. If $q$ is odd, the polarity is called orthogonal polarity. If $q$ is even, the corresponding polarity is called a pseudo-polarity. We will not go into depth on this (see [42] for details), but seen from a combinatorial point of view, every two orthogonal polarities are equivalent as are every two pseudo-polarities. This implies that we are free to choose the coordinate system and thus the matrix defining the polarity so that the absolute points can be easily described. When we speak about the polarity of $\mathbb{P}G(2,q)$, we mean either an orthogonal polarity or a pseudo-polarity, depending on the parity of $q$.

When $q$ is even, we have a line of absolute points and there is not much more to say, so from now on we assume that $q$ is odd. Let $\text{Abs}(q)$ denote the set of absolute points of the orthogonal polarity. From the two theorems above, we will deduce some properties of the structure that this polarity brings to $\mathbb{P}G(2,q)$.

Take a point $x \in \text{Abs}(q)$, then the line $x^\perp$ is absolute, so we know that $x$ is the only absolute point on this line. On the other hand, the $q$ lines through $x$, which are not $x^\perp$, intersect $\text{Abs}(q)$ in exactly one other point. Furthermore, we know that a line intersects $\text{Abs}(q)$ in at most 2 points. The only lines that have one point in common are the absolute lines, also called the tangent lines. If a line contains no absolute points we call it external, if it contains two it is called a secant. We already have $q+1$
tangent lines and we can count the number of secant lines by counting in two ways the triples \((L, x, y)\) where \(L = xy, x, y \in \text{Abs}(q)\). This leads us to \(q(q + 1)/2\) secants and consequently, the \(q(q - 1)/2\) remaining lines are external. In this way we have divided the lines into three classes. We can do the same for the points.

Take a non-absolute point \(x\). If the point lies on at least one tangent line, then \(x\) is called an **external point**, otherwise it is called an **internal point**. Denote the set of external points and internal points by \(\text{Ext}(q)\) and \(\text{Int}(q)\) respectively. Let \(x \in \text{Ext}(q)\). Then \(x^\perp\) contains at least one absolute point, but is not absolute itself. Therefore, it contains exactly two absolute points and is therefore a secant. Conversely, if \(L\) is a secant, then \(L^\perp\) lies on exactly two tangent lines and is thus an external point. We conclude that the polarity maps external points to secants and vice versa. As it also maps absolute points to tangent lines, it follows that it maps internal points to external lines. Although the terms might seem a bit confusing, they have a solid background. We show why using Figure 2.1.

![Figure 2.1: A red external point, a green internal point and blue absolute points.](image)

**Intermezzo.** The terminology **pole** and **polar** originate from the 19th century. Here the pole (a point) and the polar (a line) are related to each other with respect to a certain conic section in \(\mathbb{R}^2\). A conic has the property that a line intersects it in at most two points. If the pole lies on the conic, then its polar is the tangent line in this point to the conic. If it lies outside of the conic, then we can draw two tangents through the point and consider the line through the two tangent points; this is its corresponding polar, see the construction of the red line in Figure 2.1. If the pole lies inside the conic, then its polar line will not intersect the conic, this is the green point and line. This gives the following correspondence of terminology and its origin.

- **absolute point** \(\leftrightarrow\) pole on the conic
- **external point** \(\leftrightarrow\) pole outside of the conic
- **internal point** \(\leftrightarrow\) pole inside the conic
- **tangent line** \(\leftrightarrow\) ‘touching’ line (Latin)
- **secant line** \(\leftrightarrow\) ‘cutting’ line (Latin)
- **external line** \(\leftrightarrow\) polar skew to the conic.

The term ‘polar image’ is also derived from this context. We hope that this clarifies the terms and definitions we made and helps remembering them. **End of intermezzo.**
Now we know that the points and lines are partitioned into three classes, we want to know how many lines of each class go through a fixed point and vice versa. We will divide our discussion into three cases.

1. \(x \in \text{Abs}(q)\),
2. \(x \in \text{Ext}(q)\),
3. \(x \in \text{Int}(q)\).

We determine how many lines of each class go through \(x\) in all 3 cases. Dually, we have then also determined how many points of each class lie on a fixed line.

1. We already saw that \(x\) lies on exactly one absolute line and \(q\) secants.
2. As \(x\) lies on exactly two tangent lines, the other lines containing absolute points are secants. As \(|\text{Abs}(q)| = q + 1\), it follows that \(x\) is incident with \((q - 1)/2\) secants and \((q - 1)/2\) external lines.
3. There are no absolute lines through \(x\), so the lines incident with \(x\) containing points of \(\text{Abs}(q)\) are again secants. This implies that \(x\) is incident with \((q + 1)/2\) secants and \((q + 1)/2\) external lines.

Dually, we find

1. An absolute line contains one absolute point and \(q\) external points.
2. A secant line contains two absolute points, \((q - 1)/2\) external points and \((q - 1)/2\) internal points.
3. An external line contains \((q + 1)/2\) external points and \((q + 1)/2\) internal points.

We can summarize all this information in Table 2.1.

<table>
<thead>
<tr>
<th>(x^\perp \cap )</th>
<th>(\text{Abs}(q))</th>
<th>(\text{Ext}(q))</th>
<th>(\text{Int}(q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x \in \text{Abs}(q))</td>
<td>1</td>
<td>(q)</td>
<td>0</td>
</tr>
<tr>
<td>(x \in \text{Ext}(q))</td>
<td>2</td>
<td>(q - 1)</td>
<td>(q - 1)</td>
</tr>
<tr>
<td>(x \in \text{Int}(q))</td>
<td>0</td>
<td>(q + 1)</td>
<td>(q + 1)</td>
</tr>
</tbody>
</table>

Table 2.1: The structure in \(x^\perp\).

We will come back to this table in Section 2.2.

To conclude this section, we return to collineations. While the polarity adds structure, it takes away some of the symmetry that a projective plane has: not all points behave the same anymore. Therefore, we cannot hope that every collineation preserves the polarity. However, there exists a subgroup of \(\text{PGL}(3, q)\) that does so. This group is the \(\textbf{orthogonal group} \ PGO(3, q)\). This means that if \(\alpha \in PGO(3, q)\), we have \(\alpha(x^\perp) = \alpha(x)^\perp\). We have the following standard result.

**Theorem 2.13.** The subgroup \(\text{PGO}(3, q)\) is isomorphic to \(\text{PGL}(2, q)\).

We can even make this isomorphism explicit if the polarity is given. We will do so a few times in Section 2.2.3. This group has three orbits on \(\text{PG}(2, q)\) when \(q\) is odd: \(\text{Abs}(q)\), \(\text{Ext}(q)\) and \(\text{Int}(q)\). This means we can map any absolute point to any absolute point, while still preserving the polarity, and similarly for the other two classes of points.

With all this theory developed, we are set to take on this chapter.
2.1 History

We can immediately find an upper bound for \( \text{ex}(n, C_4) \) by the following reasoning. Let \( G \) be a \( C_4 \)-free graph on \( n \) vertices. In this chapter we always assume that \( n \geq 4 \). Then every two vertices in \( G \) have at most one common neighbour. Using this, we can double count triples of vertices \( (x, y, a) \) such that \( x \sim a \sim y \), i.e., the three vertices form a path \( P_3 \). Let \( N \) be the number of these triples. First, we can choose \( x \) and \( y \) and use that they have at most one common neighbour, so at most one candidate for \( a \):

\[ N \leq n(n-1). \]

On the other hand, we can start from the vertex \( a \) to find

\[ N = \sum_{a \in V(G)} d(a)(d(a) - 1) = \sum_{a \in V(G)} (d(a)^2 - d(a)) \geq \frac{4e(G)^2}{n} - 2e(G), \]

where in the last inequality we used Lemma 1.6. Combining both inequalities, we see that

\[ \frac{4e(G)^2}{n} - 2e(G) \leq n(n-1), \]

with equality if and only if every two vertices have exactly one common neighbour and \( G \) is regular. We can solve this equation for \( e(G) \) and find

\[ e(G) \leq \frac{n}{4} \sqrt{4n-3} + \frac{n}{4}. \]  

(2.1)

Can we achieve equality here? In order to have so, we need a regular graph where every two vertices have exactly one common neighbour. This leads to following definition.

Definition 2.14. A strongly regular graph with parameters \((v, k, \lambda, \mu)\), denoted as a \((v, k, \lambda, \mu)\)-srg, is a \( k \)-regular graph on \( v \) vertices such that every two adjacent vertices have \( \lambda \) common neighbours and every two non-adjacent vertices have \( \mu \) common neighbours.

We see that in order to have equality, we are looking for a strongly regular graph with parameters \((n, k, 1, 1)\). The question is: does this graph exist? It turns out the answer is negative, as proved by Erdős, Rényi and Sós in 1966 [29]. Their result is more commonly known as the friendship theorem and is also featured in *Proofs from the Book* [3]. Recall the graph \( F_{2k+1} \), the \( k \)-fan, we defined in the previous chapter.

Theorem 2.15. Let \( G \) be a graph such that every two vertices have exactly one common neighbour, then \( v(G) = n \) is odd and \( G = F_n \).

In particular, \( G \) cannot be regular and hence, a \((n, k, 1, 1)\)-srg does not exist. Therefore the bound we found above is actually a strict inequality. One can ask if it gives the correct order of magnitude, and it turns out it does. The graph \( F_n \), where \( n = 2k + 1 \) for some \( k \geq 1 \), has \( 3k \) edges and hence cannot give the correct magnitude. A better example is the incidence graph of the projective plane, this graph is \( C_4 \)-free on \( 2(q^2 + q + 1) = n \) vertices and has \( (q^2 + q + 1)(q+1) \approx \frac{1}{2}n^{3/2} \) edges. This shows that (after applying a density argument, which we will make more precise later on)

\[ \text{ex}(n, C_4) \approx \frac{1}{2}n^{3/2}. \]

This was already known in 1966 when Brown [14] and Erdős, Rényi and Sós [29] gave constructions showing that (2.1) is asymptotically sharp. In 1989, Clapham, Flockhart and Sheehan [17] computed \( \text{ex}(n, C_4) \) when \( n \leq 21 \). This result was extended by Yuansheng and Rowlinson [80] to \( n \leq 31 \) in 1992. We plot these exact values against the function \( f(x) = \frac{1}{2}x^{3/2} \) in Figure 2.2.
The quadrilateral

Figure 2.2: The green line is the function $f(x)$, red dots are values for which projective planes exist.

It seems that exact values slope upwards until they reach a value for which a projective plane exists and then slightly drop off again; this is most clear when $n = 21$. This gives an indication that values for which projective planes exist have relatively more edges than others and hints to the importance of projective planes in this chapter.

Clearly, the incidence graph which we constructed above can not be extremal, as we can add one edge in one part and find a $C_4$-free graph with more edges. But among $C_4$-free bipartite graphs where both parts have $q^2 + q + 1$ vertices, this graph could be a good candidate to be extremal. It turns out that it is indeed extremal in this sense. This is a question that is known as a special case of the Zarankiewicz problem posed by Kazimierz Zarankiewicz in 1951 \([81]\). This special case asks for the maximum number of 1’s in an $m \times n$ 0-1-matrix such that it contains no $2 \times 2$ minor consisting of all 1’s. This maximum is denoted by $Z(m, n, 2, 2)$. By 0-1-matrix we mean a matrix with entries either 0 or 1. Take such a 0-1-matrix $M$ containing no $2 \times 2$-minor of 1’s. Then consider a bipartite graph with parts of size $m$ and $n$, where the rows are indexed by the first part and the columns by the second, and two vertices $v_i, w_j$ belonging to different parts are adjacent if and only if the $(i, j)$-th entry is 1. Then this bipartite graph is $C_4$-free. Therefore we can translate the Zarankiewicz problem to extremal graph theory. Although this problem is concerned with bipartite graphs, it can be used to obtain results regarding the Turán problem. Define the bipartite Turán number for $C_4$ as

$$ex_B(m, n, C_4) = \max \{e(G) \mid G \text{ is bipartite with parts of sizes } m \text{ and } n, \ C_4 \not\subseteq G\}.$$ 

We see that $Z(m, n, 2, 2) = ex_B(m, n, C_4)$, so it might appear to be a trivial definition, but in Chapter 3, we will replace $C_4$ by other graphs, and then equality will no longer hold. Using this notation we have the following inequalities

$$2ex(n, C_4) \leq ex_B(n, n, C_4) \leq ex(2n, C_4) \leq 2ex_B(2n, n, C_4).$$ \(2.2\)

To see these, take a graph $G \in EX(n, C_4)$ and construct a bipartite graph by taking as vertex sets two disjoint copies of $V(G)$ and two vertices from different copies are adjacent if their corresponding vertices were adjacent in $G$. Then this graph is also $C_4$-free. This is the first inequality. The second inequality is immediate and the last one is due to a lemma by Erdős.

**Lemma 2.16.** Every graph $G$ contains a bipartite graph $H$ such that $e(H) \geq \frac{1}{2}e(G)$. 

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Proof. Partition $V(G)$ into two parts, say $X$ and $Y$, and let $H$ be the bipartite graph on these parts, so $E(H) = E(X,Y) \subseteq E(G)$. Let $x \in V(G)$, then denote by $d_G(x)$ its degree in $G$ and by $d_H(x)$ its degree in $H$. If for any vertex $x$, $d_H(x) < \frac{d_G(x)}{2}$, we can move $x$ to the other part such that now $d_H(x) \geq \frac{d_G(x)}{2}$. We can keep switching vertices from one part to another in this way, and each time the number of edges in $H$ increases. Therefore, this process must eventually stop. After this process, we see that $d_H(x) \geq \frac{d_G(x)}{2}$ for every vertex $x$ and hence $e(H) \geq \frac{1}{2}e(G)$. \hfill $\square$

Clearly if $H \subseteq G$ is a bipartite subgraph and $v(G) = 2n$, then $H \subseteq K_{2n,n}$.

We can see from the inequalities in (2.2) that these functions grow very similar, up to constants: if we know the order of magnitude of one, we know the other. Therefore, also $\text{ex}_B(n,n, C_4) = \Theta(n^{3/2})$. Actually, it turns out that determining exact values for $\text{ex}_B(n,n, C_4)$ is a lot easier than for $\text{ex}(n, C_4)$.

We will show the most recent results due to Damásdi, Héger and Szőnyi [20], without proofs.

Theorem 2.17. Let $q \geq 15$ and $c \leq q/2$, then

$$\text{ex}_B(q^2 + q - c, q^2 + q + 1, C_4) \leq (q^2 + q + 1 - c)(q + 1),$$

and equality holds if and only if a projective plane of order $q$ exists.

We see for example that when $c = 0$, we find that the incidence graph of a projective plane is indeed extremal. When $c \neq 0$, we can delete $c$ points from the incidence graph and find equality again. It can be shown that these are precisely the extremal graphs. They have found many more exact results, whenever a projective plane of order $q$ exists. The extremal graphs are again induced subgraphs of the incidence graph (obtained by deleting rows or columns from the incidence matrix).

$$\text{ex}_B(q^2 + c, q^2 + q, C_4) = q^2(q + 1) + cq \quad (0 \leq c \leq q + 1)$$

$$\text{ex}_B(q^2 - q + c, q^2 + q - 1, C_4) = (q^2 - q)(q + 1) + cq \quad (0 \leq c \leq 2q)$$

$$\text{ex}_B(q^2 - 2q + 1 + c, q^2 + q - 2, C_4) = (q^2 - 2q + 1)(q + 1) + cq \quad (0 \leq c \leq 3(q - 1)).$$

Above results are in stark contrast to when we do not restrict ourselves to bipartite graphs. Exact values of $\text{ex}(n, C_4)$ are a lot rarer and harder to determine, only for particular values of $n$ this has been done. For example, Füredi determined the exact value when $n = q^2 + q + 1$, if and only if a projective plane of order $q$ possessing a polarity exists. This existence criterion for exact values keeps on returning and shows for example that computing $\text{ex}(157, C_4)$ and determining whether a projective plane of order 12 exists are nearly equivalent. Finding these values is hence highly non-trivial.

Now when $n = q^2 + q + 1$, where $q$ is a prime power from now on, inequality (2.1) shows that

$$\text{ex}(q^2 + q + 1, C_4) < \frac{1}{2}(q^2 + q + 1)(q + 1).$$

The construction by Brown and Erdős, Rényi and Sós we mentioned earlier, gives a lower bound for $\text{ex}(q^2 + q + 1, C_4)$.

Construction 2.18. We construct the polarity graph, also known as the Erdős-Rényi graph and commonly denoted by $ER_n[1]$. Consider the projective plane $PG(2, q)$, where $q$ is a prime power, and a polarity $\beta$ of the plane with $q + 1$ absolute points. The vertices of this graph are the points of $PG(2, q)$ (or equivalently, the lines of $PG(2, q)$) and two vertices $x, y$ are adjacent if and only if $x \in y^\perp$, or equivalently $y \in x^\perp$. One can check that this graph is $C_4$-free and has $\frac{1}{2}(q^2 + q + 1)(q + 1) - (q + 1) = \frac{1}{2}q(q + 1)^2$ edges. This is because every vertex $x$ is adjacent to the $q + 1$ vertices of the line $x^\perp$, but then absolute points would create loops, so we need to delete these $q + 1$ loops.

1. It appears that this graph was constructed earlier in 1963 by Erdős and Rényi, therefore, it is named after only these two authors. However, the paper in which they do so is written in Hungarian, so we cannot verify this.
For example, the graph shown on the cover is $ER_3$. The colours of the vertices have a meaning that will be explained in a later section. It has to be mentioned that there exist other polarity graphs, where the polarity is not necessarily that of a projective plane or of projective planes that are not Desarguesian. However, in the context of $ex(n, C_4)$, the Erdős-Rényi graph (coming from the Desarguesian projective plane $PG(2, q)$) is the only polarity graph under consideration, and as such, we will talk about the polarity graph.

The polarity graph shows that
$$\frac{1}{2}q(q+1)^2 \leq ex(q^2 + q + 1, C_4) < (q^2 + q + 1)(q + 1).$$

Füredi’s contribution is showing that when $q \geq 15$, we also have $ex(q^2 + q + 1, C_4) \leq \frac{1}{2}q(q+1)^2$, which gives equality when $q$ is a prime power, and that the polarity graph is the only extremal graph $[37]$. However, in the following theorem, $q$ does not necessarily has to be a prime power and as thus, we explicitly add it in the statement

**Theorem 2.19.** Let $q \geq 2$ be an even integer or $q \geq 15$, then
$$ex(q^2 + q + 1, C_4) \leq \frac{1}{2}q(q+1)^2.$$  

Füredi first proved this for even $q$ and then was able to complete it for all integers $q$. The polarity graph has a different structure depending on whether $q$ is even or odd (now $q$ is a prime power as projective planes are only known to exist for prime powers), and when $q$ is even, it is easier to describe. This is the reason why the proof needs to be split up in these cases and probably why the $q$ even case is easier.

**Theorem 2.20.** Let $q \geq 15$ be a prime power, then
$$ex(q^2 + q + 1, C_4) = \frac{1}{2}q(q+1)^2.$$  

Moreover, the extremal graphs are exactly the polarity graphs.

As for $q = 2, 3$, there are several extremal graphs, we cannot leave out the condition that $q > q_0$ for some $q_0$. However, for $q = 4, 5$, the extremal graphs were constructed $[17]$ $[80]$ and the unique extremal graph is again the polarity graph. It is an open problem whether this also happens when $q = 7, 9, 11, 13$ (the case $q = 8$ has been proved by Füredi).

We now show the promised density argument, where constructions for prime powers give the asymptotic growth for $ex(n, C_4)$.

**Corollary 2.21.**
$$ex(n, C_4) = \frac{1}{2}n^{3/2} + o(n^{3/2}).$$

**Proof.** To find a graph on $n$ vertices with approximately $\frac{1}{2}n^{3/2}$ edges, we search $r$ such that $n - r = q^2 + q + 1$ for some prime power $q$, where $r$ is minimised. Then we can construct the polarity graph on these $q^2 + q + 1$ vertices and add $r$ vertices with degree 0. As $(q + 1)^2 > n - r$, this gives a graph on $n$ vertices with at least
$$\frac{1}{2}q^3 > \frac{1}{2}((n - r)^{1/2} - 1)^3 > \frac{1}{2}(n - r)^{3/2} - \frac{3}{2}(n - r)$$
edges. To be able to estimate this, we need to have some control on $r$: how small can this be chosen? Bombieri’s theorem on the distribution of primes $[11]$ shows that $r \leq c\sqrt{n \log n}$ for some constant $c > 0$. Therefore, for every $\epsilon > 0$, there exists a value $n_0$ such that when $n \geq n_0$ we have
$$\frac{1}{2}(n - r)^{3/2} - \frac{3}{2}(n - r) > \frac{1}{2}(1 - \epsilon)n^{3/2}.$$  

The upper bound follows from (2.1).
We will show Theorem 2.19 for even \( q \) in several steps. The proof for odd \( q \) requires a bit more machinery and results on projective planes, we will omit as such. The next lemma holds for all \( q \geq 3 \) and is the first important step in the proof.

**Lemma 2.22.** Let \( G \) be a \( C_4 \)-free graph on \( q^2 + q + 1 \) vertices, where \( q \geq 3 \). If \( \Delta(G) \geq q + 2 \) then \( e(G) \leq \frac{1}{2} q(q + 1)^2 \).

**Proof.** Let \( V(G) = \{v_1, \ldots, v_n\} \), where \( n = q^2 + q + 1 \). Without loss of generality, suppose \( v_1 \) is a vertex with \( d(v_1) = \Delta(G) = d \geq q + 2 \). We will reason in a similar way as in (2.1), but now the endpoints of the paths are not allowed to be in \( N(v_1) \). As every \( v_i \), \( 2 \leq i \leq n \), has at most one common neighbour with \( v_1 \), we find that \( d(v_i) = |\{w \sim v_i \mid w \notin N(v_1)\}| \geq d(v_i) - 1 \). Again we can count triples of vertices \((x, y, z)\), where \( x \sim y \sim z \) and \( x, z \notin N(v_1) \). Denote this number of triples by \( N \).

On the one hand,
\[
N \leq (n - d)(n - d - 1),
\]
as every two vertices have at most one common neighbour. On the other hand,
\[
N = \sum_{i=2}^{n} \tilde{d}(v_i)(\tilde{d}(v_i) - 1) \geq \sum_{i=2}^{n} (d(v_i) - 1)(d(v_i) - 2).
\]
Combining the two and applying Lemma 1.6
\[
(\frac{2}{n - 1}) \geq \frac{2e(G) - d^2}{n - 1} - 3(2e(G) - d) + 2(n - 1).
\]
Assume that \( e(G) > \frac{1}{2} q(q + 1)^2 = \frac{1}{2} (n - 1)(q + 1) \), so \( 2e(G) \geq (n - 1)(q + 1) + 2 \), then the above becomes
\[
(n - 1)(n - d)(n - d - 1) \geq [(n - 1)(q + 1) + 2 - d]^2 - 3[(n - 1)(q + 1) + 2 - d] + 2(n - 1)^2
\]
\[
\geq [(n - 1)(q + 1) + 2 - d - (n - 1)][(n - 1)(q + 1) + 2 - d - 2(n - 1)]
\]
\[
\geq [(n - 1)q + 2 - d][(n - 1)(q - 1) + 2 - d],
\]
where the second inequality follows from \( a^2 - 3ab + 2b^2 = (a - b)(a - 2b) \). However, as \( d \geq q + 2 \),
\[
n - 3 = (q + 2)(q - 1) \leq d(q - 1) \quad \Rightarrow \quad q(n - d - 1) \leq (n - 1)(q - 1) + 2 - d
\]
\[
n + q - 2 = q(q + 2) - 1 < qd \quad \Rightarrow \quad (q + 1)(n - d) < (n - 1)q + 2 - d,
\]
which, after multiplication of both sides, contradicts (2.3). \( \square \)

**Proof of Theorem 2.19** \( q \) even. Let \( G \) be a \( C_4 \)-free graph on \( q^2 + q + 1 \) vertices, then by the previous lemma, \( \Delta(G) \leq q + 1 \). If \( \Delta(G) = q \), then \( 2e(G) \leq q(q^2 + q + 1) < q(q + 1)^2 \). Therefore, we can assume that \( \Delta(G) = q + 1 \). Let \( v \) be a vertex of maximum degree and consider the induced subgraph on \( N(v) \), denoted by \( H \). Every vertex has degree at most one as otherwise we would have a vertex having more than one common neighbour with \( v \). Therefore, \( e(H) \leq d(v)/2 = (q + 1)/2 \) and, because \( q \) is even, this reduces to \( e(H) \leq q/2 \). Now
\[
\sum_{w \in N(v)} d(w) \leq d(v) + 2e(H) + (v(G) - (q + 1) - 1)
\]
\[
\leq (q + 1)d(v) - 1,
\]
as every vertex outside of \( N(v) \) has at most one common neighbour with \( v \) and hence is adjacent to at most one vertex from \( N(v) \). We conclude that there is at least one vertex of degree at most \( q \) adjacent to \( v \), or as \( v \) was arbitrary, to every vertex of maximum degree \( q + 1 \), there is at least one vertex adjacent
of degree at most $q$. Denote the number of vertices with degree at most $q$ by $a$, then we can double count the number of pairs $(v, w)$ such that $d(v) = q + 1, d(w) \leq q, v \sim w$. This leads to the equation

$$q^2 + q + 1 - a \leq aq,$$

which shows that $q < a$ or equivalently, $q + 1 \leq a$. Therefore we have at least $q + 1$ vertices of degree at most $q$ and hence the number of edges is at most

$$2e(G) \leq q^2(q + 1) + (q + 1)q = q(q + 1)^2.$$

To conclude, starting from the extremal graphs in $EX_B(n, n, C_4)$, we managed to construct the extremal graphs in $EX(n, C_4)$ by identifying in a clever way the vertices of both parts. We could not avoid losing out on $q + 1$ absolute points in this way, but the construction is the best possible. With this result available to us, we could try to explore related problems. For example, by deleting a vertex with degree $q$ from the polarity graph, we find that

$$q(q + 1)^2 - q \leq \text{ex}(q^2 + q, C_4).$$

As the polarity graph was extremal when $n = q^2 + q + 1$, we could hope that this property is kept upon removing one vertex of lowest degree. Firke, Kosek, Nash and Williford managed to prove this when $q$ is an even integer [35].

**Theorem 2.23.** Let $q \geq 2$ be an even integer, then

$$\text{ex}(q^2 + q, C_4) \leq \frac{1}{2}q(q + 1)^2 - q.$$

**Theorem 2.24.** Let $q$ be an even prime power, then

$$\text{ex}(q^2 + q, C_4) = \frac{1}{2}q(q + 1)^2 - q.$$

Moreover, for all but finitely many $q$, the extremal graphs are exactly the polarity graphs with a vertex of degree $q$ deleted.

The proof of both theorems is rather long, but we can remark that the first step in proving Theorem 2.23 remains almost the same as in Theorem 2.19, proving that for a $C_4$-free graph $G$ on $q^2 + q$ vertices holds that if $\Delta(G) \geq q + 3$, then $2e(G) \leq q(q + 1)^2 - q$.

In this context, the following conjecture of McCuaig fits nicely. The previous result shows some stability of $ER_q$ when $q$ is an even prime power. The conjecture states that this stability is much stronger than just deleting one vertex.

**Conjecture 2.25.** If $G \in EX(n, C_4)$, then $G$ is an induced subgraph of $ER_q$ for some $q$.

This conjecture has only been verified for $n \leq 31$ and the case when $n = q^2 + q$, $q$ an even prime power. At this moment, solving this conjecture seems out of reach.

The problem of determining $\text{ex}(n, C_4)$ for other values of $n$ has recently received some attention. The polarity graph is used in the majority of this research and our knowledge about it is rapidly growing. In [1], Abreu, Balbuena and Labbate show some constructions of $C_4$-free graphs by deleting vertices from polarity graphs in a systematic way using the language of matrices. We mention a few of their results, some of which we can reprove in a geometrical way when we look a bit closer at the structure of $ER_q$ in the next section.
Theorem 2.26. Let $q = p^h \geq 3$, $p$ prime, $h \geq 1$,

1. $ER_2$ is a subgraph of $ER_q$.

2. $ER_{t^h}$ is a subgraph of $ER_q$ for every $t$ that divides $h$.

3. In particular, when $h$ is even, then $ER_{\sqrt{q}}$ is a subgraph of $ER_q$

As we deleted a vertex from $ER_q$ to find a lower bound for $ex(q^2 + q, C_4)$, we can delete a subgraph from the polarity graph to find new $C_4$-free graphs. Recall that by deleting a subgraph $H$ from a graph $G$ we mean we delete the vertices of $H$ from $G$. When deleting a subgraph $H$ from $ER_q$, the number of edges becomes

$$e(ER_q - H) = e(ER_q) - \sum_{v \in V(H)} d(v) + e(H),$$

which is the original number of edges minus all edges adjacent to deleted vertices but corrected by $e(H)$, as we have subtracted those twice in the second term. As $ER_q$ is nearly regular (every vertex has degree $q$ or $q+1$), the second term is more or less determined by $v(H)$, while $e(H)$ depends on the structure of $H$. Therefore, when searching for dense induced subgraphs of $ER_q$, we need to find dense subgraphs to delete. This might seem a bit counter-intuitive at first, but the above computation should be convincing and gives the reason why. To summarise, in the spirit of McCuaig’s conjecture, when looking to give lower bounds for $ex(q^2 + q + 1 - a, C_4)$, we look for the densest subgraph possible on $a$ vertices and delete it from $ER_q$. This is also what Abreu, Balbuena and Labbate have done. They found following results.

**Proposition 2.27.** Let $q$ be a prime power,

1. $ex(q^2 - \sqrt{q}, C_4) \geq \frac{1}{2}q(q^2 - 1) - \frac{1}{2}\sqrt{q}(q - 1)$ if $q$ is a square,

2. $ex(q^2 - q - 2, C_4) \geq \begin{cases} 
\frac{1}{2}q(q^2 - 1) & \text{if } q \text{ is odd,} \\
\frac{1}{2}q^3 - q^2 & \text{if } q \text{ is even.}
\end{cases}$

Remark that the first one follows immediately by deleting $ER_{\sqrt{q}}$ from $ER_q$. They conjectured that these bounds were the best possible. For the first one, this is still an open question, while the second one has been improved by Tait and Timmons [71].

**Proposition 2.28.** $ex(q^2 - q - 2, C_4) \geq \frac{1}{2}q^3 - q^2 + \frac{3}{2}q - O(\sqrt{q})$.

In [70] they also showed a lower bound for $ex(q^2 - 1, C_4)$.

**Proposition 2.29.** Let $q = p^{2k}$ for some prime $p$ and $k \geq 1$, then

$$ex(q^2 - 1, C_4) \geq \frac{1}{2}q^3 + \frac{1}{2}q^{3/2} - \frac{3}{2}q + \frac{1}{2}\sqrt{q} - 2.$$

These are all of the most current results regarding $ex(n, C_4)$. The hardest part seems to be finding good upper bounds.

**Question 2.30.** Determine $ex(q^2 + q + 1, C_4)$ and $EX(q^2 + q + 1, C_4)$ for $5 \leq q \leq 14$. Extend the computation for $ex(n, C_4)$ to $n > 31$. 

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As we already remarked earlier, computing the values of ex(157, C_4) and ex(241, C_4) is equivalent to proving the (non)-existence of projective planes of order 12 and 15 with a polarity respectively. Possibly, these projective planes exist, but do not possess polarities. So we cannot immediately conclude their non-existence. A way to avoid this obstacle is by computing ex(B(q^2 + q + 1, q^2 + q + 1, C_4)) for these values of q. The disadvantage is that these graphs have twice the number of vertices, so computations might be harder.

**Question 2.31.** Does the stability result of Theorem 2.23 also hold when q is an odd prime power and are the extremal graphs again subgraphs of ER_q?

**Question 2.32.** Find good upper bounds for ex(n, C_4) when n is one of the values in Propositions 2.27, 2.28 and 2.29

### 2.2 The polarity graph

In this section, we take a closer look at the polarity graph ER_q. As before, q will always be a prime power here. Since its construction, it has appeared in several topics besides Turán-type problems. This is due to the fact that the polarity graph is extremal in many ways. For example, it has been used in Ramsey theory by Parsons \[61\] and in the degree diameter problem, see the survey by Miller and Širáň \[59\]. For small values of q, many interesting graphs appear as subgraphs of ER_q. Among those graphs are the Petersen graph, the Coxeter graph, the Tutte-Coxeter graph, the line graph of the Heawood graph, etc. All of these facts motivate a deeper look into ER_q.

We will show some immediate connections between PG(2,q) and ER_q. The first one relates to automorphisms of ER_q.

As every \( \alpha \in \text{PGO}(3, q) \) leaves the polarity invariant, it follows that \( \alpha \) is also an automorphism of ER_q. Actually, similarly as PGL(3,q) and PTL(3,q), there exists an extension PGO(3,q) of PGO(3,q) when also considering the automorphisms of GF(q). Then also every \( \alpha \in \text{PGO}(3,q) \) is an automorphism of ER_q. One can prove the converse, that every automorphism of ER_q is an element of PGO(3,q), see \[61\] for a proof.

**Proposition 2.33.** The automorphism group of ER_q is PGO(3,q).

Secondly, we constructed ER_q from PG(2,q), but we can also do the converse construction. Let \( P = V(ER_q) \),

\[
\mathcal{L} = \left\{ \begin{array}{ll}
N(x) & \text{if } x \notin \text{Abs}(q), \\
N(x) \cup \{x\} & \text{if } x \in \text{Abs}(q),
\end{array} \right\}
\]

The incidence relation is containment. Then \( (P, \mathcal{L}, 1) \) is again a projective plane, isomorphic to PG(2,q).

Now we will determine the structure of ER_q. In the next section we use this knowledge to find large independent sets. Recall that an independent set is a set of pairwise non-adjacent vertices. First we recall Baer’s result, applied to PG(2,q).

**Theorem 2.34.** Let \( \beta \) be a polarity of PG(2,q) with \( q + 1 \) absolute points.

- q is even if and only if all absolute points lie on a line;
- q is odd if and only no line contains more than two absolute points.

This shows that there is big difference in the structure of ER_q depending on whether q is even or odd. As such, we will treat both cases separately.
### 2.2.1 $q$ even

By Theorem 2.11 we know that the line $L$ containing all absolute points cannot be absolute. Therefore, $L$ cannot be the polar image of any absolute point. It follows that $L = l^\perp$, where $l$ is a point outside of $L$. Then for any absolute point $x, x \in x^\perp$ and $x \in l^\perp$, so $l \in x^\perp$. We see that the absolute line through $x$ is the line $xl$. If $L_1, \ldots, L_{q+1}$ denote the absolute lines then any non-absolute point which is not $l$ lies on exactly one absolute line. In other words, the absolute lines partition $PG(2, q) \setminus \{l\}$. We can show this in Figure 2.3.

![Figure 2.3: The structure of the points and lines under the polarity.](image)

Let $x = L_1^\perp$ be an absolute point. Then $x$ is adjacent to $l$ and the $q - 1$ other non-absolute points on the line $L_1$. Let $y$ be one of these points. So $y \sim x$, but it is not adjacent to any other point on the line $L_1$ as $L_1 \cap y^\perp = \{x\}$. However, it is adjacent to exactly one point on every line $L_2, \ldots, L_{q+1}$. This point is $L_i \cap y^\perp, 2 \leq i \leq q + 1$, and is clearly never the point $l$. We can summarize all of this in the following way.

We have three classes of points:

1. The point $l$ which is adjacent to all absolute points,

2. the absolute points, adjacent to the $q$ non-absolute points on their absolute line, among which $l$,

3. the non-absolute points (but not $l$), adjacent to one point on every absolute line, among which one absolute point. This absolute point corresponds to the unique absolute line on which the non-absolute point lies.

Therefore, we can see that the structure of $ER_q$, when $q$ is even, should look like in Figure 2.4.

![34]
The structure of $ER_q$, $q$ even.

### 2.2.2 $q$ odd

The structure of $ER_q$, $q$ odd, is a bit more intricate, and also depends on the class of $q$ modulo 4. We already know the existence of the sets $\text{Abs}(q)$, $\text{Ext}(q)$ and $\text{Int}(q)$. The only loops that appear are for absolute points, so Table 2.1 translates immediately into Table 2.2.

<table>
<thead>
<tr>
<th>neighbours in $\text{Abs}(q)$</th>
<th>$\text{Ext}(q)$</th>
<th>$\text{Int}(q)$</th>
<th>$d(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \in \text{Abs}(q)$</td>
<td>0</td>
<td>0</td>
<td>$q$</td>
</tr>
<tr>
<td>$x \in \text{Ext}(q)$</td>
<td>$\frac{q-1}{2}$</td>
<td>$\frac{q-1}{2}$</td>
<td>$q + 1$</td>
</tr>
<tr>
<td>$x \in \text{Int}(q)$</td>
<td>0</td>
<td>$\frac{q+1}{2}$</td>
<td>$q + 1$</td>
</tr>
</tbody>
</table>

Table 2.2: The top-left entry is now 0 as an absolute point can never be adjacent to another absolute point. It was a loop we deleted.

Moreover, we also know the sizes of the respective sets:

$$|\text{Abs}(q)| = q + 1, \quad |\text{Ext}(q)| = \frac{q(q + 1)}{2}, \quad |\text{Int}(q)| = \frac{q(q - 1)}{2}.$$  

For example, using this table, one can see that the induced subgraph of $ER_q$ with only vertices of $\text{Ext}(q)$, also denoted by $\text{Ext}(q)$, is a $(q - 1)/2$-regular graph on $q(q + 1)/2$ vertices, and similarly for $\text{Int}(q)$. We will discuss these subgraphs later on. As another example, we can remark that $\text{Abs}(q)$ is an independent set in $ER_q$. These numbers will be used frequently throughout the remaining part, it is therefore important to fully understand where they come from.

We are now well equipped to explore the local structure of $ER_q$, meaning that we will take a closer look at the neighbourhood of the various types of points. First we will prove two lemmas and then we will split our discussion into three cases again, the same three cases as in Section 2.0, where the point was absolute, external and internal respectively.

**Lemma 2.35.** An absolute point can never be contained in a triangle of $ER_q$.

**Proof.** Suppose $x \in \text{Abs}(q)$. If $x$ were contained in a triangle, we would need that two points $y, z$ in $N(x)$ are adjacent, meaning $z \in y^\perp$. However, $y^\perp$ is a line through $x$ intersecting $x^\perp$ only in $x$. 

\[ \text{(Figure 2.4: The structure of $ER_q$, $q$ even.)} \]
Recall that the graph $S$.

**Proof.** Suppose $\chi$. If $\gamma$ then every non-absolute point $y \in N(x)$ is contained in a triangle with $x$.

**Lemma 2.36.** If $x \notin \text{Abs}(q)$ then every non-absolute point $y \in N(x)$ is contained in a triangle with $x$.

**Proof.** Suppose $x, y \notin \text{Abs}(q)$ and $x \sim y$. Then consider the point $x^\perp \cap y^\perp$. This point cannot be $x$ or $y$ as both are not absolute, so it is a third point adjacent to both.

These two lemmas help us to determine the neighbourhood structure of any point of $ER_q$ quickly. Recall that the graph $S_n$ is the star on $n$ vertices and $F_n$ the $k$-fan when $n = 2k + 1$ is odd.

1. As $x$ is absolute, it is not contained in any triangle. It follows that $N(x)$ is an independent set, or equivalently, the induced subgraph on $N(x) \cup \{x\}$ is $S_{q+1}$.

2. The two absolute points have degree 0 in $N(x)$, while the other vertices come in pairs to form triangles with $x$. The induced subgraph on $N(x) \cup \{x\}$ is $F_q$ with two additional vertices adjacent to the center (the vertex of maximum degree).

3. All points have degree 1 in $N(x)$, the induced subgraph on $N(x) \cup \{x\}$ is $F_{q+2}$.

We see that locally, the polarity graph looks (almost) like the friendship graph in two of the three cases. So although the friendship graph was not the extremal $C_4$-free graph we hoped for, it appears frequently as a subgraph of our extremal graph.

Consider three points $x, y, z \in PG(2, q)$ such that $x, y \in z^\perp$, $x, z \in y^\perp$ and $y, z \in x^\perp$, then these points form a **self-polar triangle**. Remark that for any two non-absolute points $x, y \in PG(2, q)$, such that $x \in y^\perp$ the points $x, y, x^\perp \cap y^\perp$ form a self-polar triangle as we saw in the proof of Lemma 2.36.

The triangles appearing in $ER_q$ correspond to self-polar triangles and vice-versa, showing another part of the strong connection between $ER_q$ and $PG(2, q)$.

We saw that $ER_q$ is full of triangles, appearing in its neighbourhood structure. These triangles are not arbitrary: there can be found some structure among them. This is the content of the following known result, which is proved in a new, combinatorial way. If the points $x, y, z \in \text{Ext}(q)$ are the points of a self-polar triangle, then we say that this triangle is an $\triangle \text{EEE}$ triangle. Similarly, we can talk about $\triangle \text{EEI}$, $\triangle \text{EII}$ and $\triangle \text{III}$ triangles. We can classify the possibilities that these triangles appear in $ER_q$ in the following lemmas.

**Lemma 2.37.** If there exists $x \in \text{Ext}(q)$ such that it is contained in an $\triangle \text{EEI}$ triangle, then every external point is contained in only $\triangle \text{EEI}$ triangles.

For the proof we need the fact that for an external point $x$, the point-stabilisator subgroup $\text{Stab}_{PG(2, q)}(x)$ of $PGL(2, q)$ has three orbits on $x^\perp$: the absolute points, the external points and the internal points. Similarly, the stabiliser subgroup of an internal point has two orbits on its polar image, the external and internal points respectively. This is a standard result that can be found in [42][43][68].

**Proof.** First we prove that every triangle in $N(x) \cup \{x\}$ is an $\triangle \text{EEI}$ triangle. Suppose $x, y \in \text{Ext}(q)$ and $z \in \text{Int}(q)$ form an $\triangle \text{EEI}$ triangle. Then by the remark above, for any other point $w \in N(x) \cap \text{Ext}(q)$, there exists $\alpha \in \text{Stab}_{PG(2, q)}(x) \leq PGL(2, q)$ such that $\alpha(y) = w$. Then as $\alpha(p^\perp) = \alpha(p)^\perp$ for any point $p \in PG(2, q)$, we have

$$z = x^\perp \cap y^\perp \Rightarrow \alpha(z) = \alpha(x^\perp) \cap \alpha(y^\perp) = x^\perp \cap w^\perp.$$
The quadrilateral

We know that \( \alpha \) maps internal points to internal points, so this means that the points \( x, w \) and \( \alpha(z) \) form again an \( \triangle \)EEI triangle. As \( w \) was arbitrary, we see that every external point in \( N(x) \) matches up with an internal point to form a triangle.

For any other external point \( x' \) we can find \( \alpha \in PGL(2, q) \) such that \( \alpha(x) = x' \). Then, as \( \alpha \) is also an automorphism of \( ER_q \), we see that the vertices \( x', \alpha(y), \alpha(z) \) form an \( \triangle \)EEI triangle. Then we can repeat the argument from above and see that \( x' \) is also only contained in \( \triangle \)EEI triangles.

Therefore, if one external point is contained in a \( \triangle \)EEI triangle, all external points are contained in only \( \triangle \)EEI triangles. In this case, two adjacent internal points cannot be contained in a \( \triangle \)EII triangle (no external points are contained in an \( \triangle \)EII triangle) and hence are contained in a \( \triangle \)III triangle. On the other hand, if we find either an \( \triangle \)EEE or \( \triangle \)EII triangle, we know that we are not in the previous case and hence all triangles should be of these two types. Summarizing we find two possibilities for the triangles in \( ER_q \),

\[ ER_q \text{ contains only } \triangle \text{EEE and } \triangle \text{EII triangles OR } ER_q \text{ contains only } \triangle \text{EEI and } \triangle \text{III triangles.} \]

The question is now: given a prime power \( q \), can we know which case we are in? The answer is affirmative and follows from an easy counting argument, which is contained in the following theorem.

**Theorem 2.38.** The following are equivalent:

1. \( q \equiv 1 \pmod{4} \),
2. there exists an \( \triangle \)EEE triangle,
3. there exists no \( \triangle \)III triangle.

**Proof.** We will show that (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (1).

Suppose \( q \equiv 1 \pmod{4} \) and assume there exists no \( \triangle \)EEE triangle. Then every external point is contained in \( \triangle \)EEI triangles and therefore internal points are contained in either \( \triangle \)EII or \( \triangle \)III triangles. This means that for an internal point \( x \) the \( (q+1)/2 \) internal points in \( N(x) \) match up to form \( (q+1)/4 \) \( \triangle \)III triangles with \( x \). However, this is in contradiction with \( q \equiv 1 \pmod{4} \) as then \( (q+1)/4 \) cannot be a natural number. We conclude that there must exist an \( \triangle \)EEE triangle.

Now assume that there exists an \( \triangle \)EEE triangle. By the discussion and lemmas before, we see that every internal point is contained in only \( \triangle \)EII triangles and hence there exist no \( \triangle \)III triangles. Suppose there exist no \( \triangle \)III triangles, then every internal point is contained in \( \triangle \)EII triangles and hence external points are contained in \( \triangle \)EEE and \( \triangle \)EII triangles. Therefore, the \( (q-1)/2 \) external points of \( N(x) \), where \( x \in \text{Ext}(q) \), form \( (q-1)/4 \) triangles in \( N(x) \cup \{x\} \), which shows that \( q \equiv 1 \pmod{4} \).

**Corollary 2.39.** The following are equivalent:

1. \( q \equiv 3 \pmod{4} \),
2. there exists an \( \triangle \)III triangle,
3. there exist no \( \triangle \)EEE triangle.

We remark that all our proofs so far have been of a combinatorial nature, a line of proof we have not yet encountered in the literature, even for the previous results.

**Theorem 2.40.** The following are the types and numbers of self-polar triangles in \( ER_q \), \( q \) odd:

1. \( q \equiv 1 \pmod{4} \)

37
(a) $\triangle EEE$ $\frac{q(q^2 - 1)}{24}$
(b) $\triangle EII$ $\frac{q(q^2 - 1)}{8}$

(2) $q \equiv 3 \pmod{4}$
(a) $\triangle III$ $\frac{q(q^2 - 1)}{24}$
(b) $\triangle EEI$ $\frac{q(q^2 - 1)}{8}$

Proof. We already know when which types of triangles appear, so we only need to show how many triangles of each type we have. Suppose $q \equiv 1 \pmod{4}$. To find a $\triangle EEE$ triangle, we take an external point and one of the $\frac{(q - 1)}{4}$ $\triangle EEE$ triangles it is contained in. We could have found this triangle in three ways, from each point of the triangle. Therefore we have

$$3\# = \frac{q(q + 1)q - 1}{4},$$

where $\#$ denotes the number of $\triangle EEE$ triangles. In a similar way, one can compute the other numbers.

As an exercise, one can check all the previous results on the graph $ER_3$ in Figure 2.5, which is also the graph on the cover. The three colours represent the three classes of vertices.

As we already saw, every external point is adjacent to $(q - 1)/2$ other external points and similarly, every internal point is adjacent to $(q + 1)/2$ internal points. It seems interesting to look at the subgraphs of $ER_q$ induced by Ext$(q)$ and Int$(q)$, as they give regular graphs. These graphs, which we will also denote by Ext$(q)$ and Int$(q)$, have been studied by many authors, among which Parsons [62] was one of the first. He showed for example the following result. The girth of a graph that is not a forest, is the length of the smallest cycle of the graph.
Theorem 2.41. Let Ext(q) and Int(q) be defined as before, then we have the following two cases:

(1) \( q \equiv 1 \mod 4 \)

(a) \( \text{Ext}(q) \) is a \((q-1)/2\)-regular graph of girth 3, if \( q > 5 \), and

(b) \( \text{Int}(q) \) is a \((q+1)/2\)-regular graph of girth 5.

(2) \( q \equiv 3 \mod 4 \)

(a) \( \text{Ext}(q) \) is a \((q-1)/2\)-regular graph of girth 5, if \( q > 7 \), and

(b) \( \text{Int}(q) \) is a \((q+1)/2\)-regular graph of girth 3.

One can see in Figure 2.5 that \( \text{Ext}(3) \) consists of three independent edges, while \( \text{Ext}(5) \) is the disjoint union of 5 triangles and \( \text{Ext}(7) \) is the Coxeter graph which has girth 7 [19]. Unfortunately, with only our combinatorial and graph theoretical methods, we can show everything but statement (2.a). A proof of this, using more algebra can be found in [62, 79].

Proof. Cases (1.a) and (2.b) are the result of our discussion before. We will show (1.b).

Suppose \( q \equiv 1 \mod 4 \) and \( \text{Int}(q) \) has no cycles of length 5, then it has girth at least 6. Take any edge \( xy \in E(\text{Int}(q)) \). We will perform a breadth-first search starting from this edge. The vertices \( x \) and \( y \) are at level 0. At level 1 we find the sets \( N(x) \setminus \{y\} \) and \( N(y) \setminus \{x\} \), which are disjoint independent sets, as otherwise we would find a triangle. For level 2, take any two vertices \( v, w \) in \( (N(x) \cup N(y)) \setminus \{x, y\} \), then the intersection of \( N(v) \) and \( N(w) \) is empty, \( x \) or \( y \), as otherwise we would find a cycle of length 4 or 5. In this way, we see that if the girth is at least 6, then we can sum up the sizes of all these sets of vertices to find

\[
2 + 2 \frac{q - 1}{2} + 2 \frac{(q-1)^2}{4} = \frac{q^2 + 3q - 1}{2} = v(\text{Int}(q)),
\]

which is a contradiction. Therefore, as \( \text{Int}(q) \) has no cycles of length 3 or 4 and is clearly not a tree, it should have a cycle of length 5.

The same argument cannot be applied to \( \text{Ext}(q) \). The reason is that \( \text{Int}(q) \) is a denser graph: it has less vertices while the vertices have a higher degree.

We see that level 1 consists of vertices adjacent to either \( x \) or \( y \), denote these vertices at level 1 by \( N_1(x) \) and \( N_1(y) \) respectively. It follows that \( N_1(x) = N(x) \setminus \{y\} \). Similarly, we can define the vertices at level 2 under \( x \) or \( y \) by \( N_2(x) \) and \( N_2(y) \) respectively. A picture says more than a 1000 words, so we refer to Figure 2.6 if these concepts are not completely clear.

Now that we have figured out the structure of \( ER_q \) for any \( q \), we can give the promised proof of Theorem 2.26.

Proof of Theorem 2.26. Let \( q = p^h \geq 3 \), \( p \) prime and \( h \geq 1 \).

1. \( ER_2 \) is a subgraph of \( ER_q \). If \( q \) is even, we can take any two absolute points \( x_1, x_2 \). Then take any \( y_1 \in x_1^+ \setminus \{x_1, y\} \) and let \( y_2 = y_1^+ \cap x_2^+ \). Then define \( y_3 = y_1^+ \cap y_2^+ \) and let \( x_3 \) be the absolute point on the line \( y_3 \). One can check that the seven points \( x_i, y_i, l \) induce a subgraph isomorphic to \( ER_2 \). If \( q \) is odd, take a self-polar triangle \( x_1, x_2, x_3 \). Then take a non-absolute line \( y_1^+ \) through \( x_1 \), distinct from \( x_2^+ \), and a non-absolute line \( y_2^+ \) through \( x_2 \) in a similar way. Denote the point of intersection \( z = y_1^+ \cap y_2^+ \), this is clearly distinct from all previous points. Define the point \( y_3 \) by \( y_3^+ = x_3 \), then the points \( x_i, y_i, z \) again induce a subgraph isomorphic to \( ER_2 \).

2. \( ER_{p^t} \) is a subgraph of \( ER_q \) for every \( t \) that divides \( h \). Take all the points with coordinates in the subfield \( GF(p^t) \) of \( GF(q) \), then this defines a projective plane of order \( p^t \) and one can restrict the polarity to this plane. In particular when \( q \) is a square, \( ER_{\sqrt{q}} \) is a subgraph of \( ER_q \).
These subgraphs have been used in various topics of graph theory because of their nice properties, too many to list them all here. We can remark for example that they have been used to determine the Ramsey number $R(C_4, S_n)$, which is the minimum number of vertices $r$ such that any graph on $r$ vertices contains either $C_4$, or its complement contains $S_n$, where $n$ is in a certain range of values $[54, 61, 82]$. They also show up in the degree-diameter problem, which is another central problem in graph theory we already mentioned [6].

2.2.3 Independence number

A final topic which has attracted some interest in the last few years is finding the independence number $\alpha(ER_q)$. Recall that an independent set of a graph is a set of vertices which are pairwise non-adjacent.

Definition 2.42. Let $G$ be a graph. The independence number $\alpha(G)$ is the size of the largest independent set in $G$.

In the case of $ER_q$, the problem of determining $\alpha(ER_q)$ is equivalent to finding the maximum number of mutually non-orthogonal vectors in $V(3, q)$. Several articles have appeared on this topic, among which [44, 60, 70, 72]. The best current upper and lower bounds are

$$\alpha(ER_q) \leq \begin{cases} 
q^{3/2} - q + \sqrt{q} + 1 & \text{if } q \text{ is an even square}, \\
q^{3/2} + \sqrt{q} + 1 & \text{otherwise}.
\end{cases}$$

$$\alpha(ER_q) \geq \begin{cases} 
q^{3/2} - q + \sqrt{q} & \text{if } q \text{ is an even square}, \\
\frac{1}{2\sqrt{3}}q^{3/2} & \text{if } q \text{ is an even non-square}, \\
\frac{1}{2}q^{3/2} + \frac{1}{2}q + 1 & \text{if } q \text{ is an odd square}, \\
\frac{120}{73}\sqrt{13}q^{3/2} & \text{if } q \text{ is an odd non-square}.
\end{cases}$$

This shows that in all cases, $\alpha(ER_q) = \Theta(q^{3/2})$. However, finding the correct constant still remains an open problem, except when $q$ is an even square. In this case, the upper and lower bound differ by
The quadrilateral

only one. We will show improvements, due to the Francesco Pavese, Leo Storme and the author, for the lower bounds in the second and third case.

**Theorem 2.43.** Let \( q \) be a prime power, then

\[
\alpha(ER_q) \geq \begin{cases} 
\frac{1}{\sqrt{2}} q^{3/2} - q + \frac{1}{\sqrt{2}} \sqrt{q} & \text{if } q \text{ is an even non-square,} \\
\frac{1}{2} q^{3/2} + q - \frac{1}{2} \sqrt{q} + 1 & \text{if } q \text{ is an odd square and } \sqrt{q} \equiv 3 \pmod{4}, \\
\frac{1}{2} q^{3/2} + \frac{3}{2} q + 1 & \text{if } q \text{ is an odd square and } \sqrt{q} \equiv 1 \pmod{4}.
\end{cases}
\]

Showing this theorem involves more geometrical tools than we have seen so far. Readers not already familiar with projective geometry might need more than one reading to fully grasp the concepts and tools used. In all three cases, we will construct an independent set as the orbit \( O \) of a certain subgroup of \( \text{PGL}(2, q) \). This allows us, when we check the mutual non-orthogonality of the points, to consider only one point \( x \) and check whether \( |x^\perp \cap O| = 0 \). The reason is that for any other point \( y \in O \) we can find \( \alpha \) in the subgroup under consideration such that \( \alpha(x) = y \). Then we have

\[
|y^\perp \cap O| = |\alpha(x)^\perp \cap O| = |\alpha(x^\perp) \cap \alpha(O)| = |\alpha(x^\perp \cap O)| = 0,
\]

as \( \alpha \) is a collineation that leaves the polarity invariant and \( \alpha(O) = O \).

We start in the case when \( q \) is an even non-square.

1. \( q \) even non-square

Let \( n \geq 1 \), a maximal arc \( A \) of degree \( n \) is a set of \((n - 1)q + n\) points of \( \text{PG}(2, q) \) such that every line meets \( A \) in 0 or \( n \) points. We will show the existence of a maximal arc \( A \) of degree \( \sqrt{q/2} \) such that for every point \( x \in A \) we have that \( |x^\perp \cap A| = 0 \). This gives us an independent set of size

\[
\frac{1}{\sqrt{2}} q^{3/2} - q + \frac{1}{\sqrt{2}} \sqrt{q},
\]

as promised. We will show this in several steps.

First we need to coordinatise, in order to do computations. Without loss of generality (as we discussed in Section 2.0), we can assume that for a point \( x = (x_0, x_1, x_2) \in \text{PG}(2, q) \) we have \( x^\perp : x_0 X_0 + x_2 X_1 + x_1 X_2 = 0 \) (one can check that this indeed defines a polarity). Then the line of absolute points \( L \) is defined by \( X_0 = 0 \) and \( L^\perp = (1, 0, 0) \). The isomorphism of \( \text{PGO}(3, q) \) with \( \text{PGL}(2, q) \) can then be explicitly given by

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \leftrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix},
\]

where \( a, b, c, d \in GF(q) \) and \( ad + bc = 1 \).

Let \( \alpha \in GF(q) \) such that \( \text{Tr}(\alpha) = 1 \), then \( X^2 + X + \alpha = 0 \) is an irreducible polynomial over \( GF(q) \) by Proposition 2.1. For any \( \lambda \in GF(q) \), consider the conic

\[
C_\lambda : \lambda X_0^2 + X_1^2 + X_2 + \alpha X_2^2 = 0.
\]

Then the sets \( \{ C_\lambda \mid \lambda \in GF(q) \} \cup \{ L \} \) partition the points of \( \text{PG}(2, q) \). Moreover, they form a pencil \( F \) in which every conic has nucleus \((1, 0, 0)\). This pencil is stabilised by the following group \( H \) of order \( q + 1 \), the orbits being the conics of the pencil,

\[
H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & ab \\ 0 & b & a + b \end{pmatrix} \bigg| a, b \in GF(q), a^2 + ab + \alpha b^2 = 1 \right\}.
\]
In [22], Denniston proved that if $A$ is an additive subgroup of $GF(q)$ of order $n$, then the set of points of $C_\lambda, \lambda \in A,$ form a maximal arc of degree $n$. This means that the key is to find an additive subgroup of order $\sqrt{q/2}$ of $GF(q)$. We will do so in the following two lemmas.

**Lemma 2.44.** Let $\lambda \in GF(q)$ with $Tr(\lambda) = 0$, then for every $x \in C_{\lambda^2}$ we have $|x^\perp \cap C_{\lambda^2}| = 0$.

**Proof.** With the notation introduced above, we see that $C_{\lambda^2} = \{(1, \lambda a, \lambda b) \mid a^2 + ab + \lambda^2 b^2 = 1\}$. Let $x = (1, \lambda, 0) \in C_{\lambda^2}$, then $x^H = C_{\lambda^2}$ by the above. Therefore, we only need to check that $|x^\perp \cap C_{\lambda^2}| = 0$. The line $x^\perp$ is defined by the equation $X_0 + \lambda X_2 = 0$. If $\lambda = 0$, then the intersection is obviously empty. So suppose $\lambda \neq 0$ and that $x^\perp$ and $C_{\lambda^2}$ have a point $(1, \lambda a, \lambda b)$ in common, for some $a, b \in GF(q)$ such that $a^2 + ab + \lambda^2 b^2 = 1$. From this, it follows that $b = 1/\lambda^2$ and there exists $c \in GF(q)$ such that $a = c/\lambda^2$ and $c$ is a solution of

$$X^2 + X + (\alpha + \lambda^4) = 0.$$

However, as

$$Tr(\alpha + \lambda^4) = Tr(\alpha) + Tr(\lambda^4) = Tr(\alpha) + Tr(\lambda) = 1,$$

we see that this equation has no solutions and therefore we find a contradiction. We conclude that the intersection is indeed empty. \hfill $\square$

**Lemma 2.45.** If $q = 2^n, n$ odd, then we can find an additive subgroup $A$ of $GF(q)$ of order $\sqrt{q/2}$ such that for every $\lambda_1, \lambda_2 \in A$, we have $Tr(\lambda_1 \lambda_2) = 0$. In particular, $Tr(\lambda_1) = Tr(\lambda_2) = 0$.

**Proof.** This is done by the identification of $GF(2^n)$ with $V(n, 2) \setminus \{0\}$. As the only non-zero constant is 1, we see that the process of constructing $PG(n - 1, 2)$ from $V(n, 2)$ is rather trivial. Therefore, we can identify $GF(2^n) \setminus \{0\}$ with $PG(n - 1, 2)$. Then the trace function

$$(x_1, x_2) \in GF(2^n) \times GF(2^n) \rightarrow Tr(x_1x_2) \in GF(2)$$

can be translated into a bilinear form on $GF(2^n)$. Moreover, by the properties of the trace function, this form is non-degenerate, symmetric and non-alternating, therefore it defines a pseudo-polarity $\perp$, as $q$ is even (see again [42]). Let $M$ be a maximal absolute subspace of $PG(n - 1, 2)$, meaning that $M^\perp = M$, and $A$ the set of elements of $GF(2^n)$ corresponding to the points of $M$. As $n$ is odd, the dimension of $M$ is $(n - 1)/2$ (see [43]) and therefore $A$ is a subset of $2^{(n-1)/2}$ elements. It is clear that $A \cup \{0\}$ is an additive subgroup and for two elements $\lambda_1, \lambda_2 \in A$, that $Tr(\lambda_1 \lambda_2) = 0$. The last statement is the special case when $\lambda_1 = \lambda_2$, then we get $Tr(\lambda_1^2) = Tr(\lambda_1) = 0$. \hfill $\square$

**Theorem 2.46.** If $q$ is an odd power of 2, then there exists a maximal arc $A$ of degree $\sqrt{q/2}$ such that for every $x \in A$, we have $|x^\perp \cap A| = 0$.

**Proof.** Let $A$ be the additive subgroup from the previous lemma and $A$ the set of points of $C_\lambda$ such that $\lambda \in A$. Then by Denniston’s result, we know that $A$ is a maximal arc of degree $\sqrt{q/2}$. Take $\lambda_1, \lambda_2 \in A$, then $Tr(\lambda_1) = Tr(\lambda_2) = Tr(\lambda_1 \lambda_2) = 0$. Similar as in Lemma [2.44] we have for $x_i = (1, \lambda_i, 0)$, that $x_i^H = C_{\lambda_i^2}$, $i = 1, 2$. We already know that $|x_1^\perp \cap C_{\lambda^2_1}| = 0$ by the same lemma, so we need to check that $|x_2^\perp \cap C_{\lambda_2^2}| = 0$ and we are done, as we can then use the reasoning from equation (2.4) and the fact that $\lambda_1$ and $\lambda_2$ are arbitrary.

The line $x_1^\perp$ is defined by the equation $X_0 + \lambda_1 X_2 = 0$, so if a point $(1, \lambda_2 a, \lambda_2 b) \in C_{\lambda^2_2}$ for some $a, b \in GF(q)$ belongs to it, then $b = 1/\lambda_1 \lambda_2$ and there exists $c \in GF(q)$ such that $a = c/\lambda_1 \lambda_2$ and $x$ is a solution of

$$X^2 + X + (\alpha + \lambda_1^2 \lambda_2^2) = 0.$$

However, as

$$Tr(\alpha + \lambda_1^2 \lambda_2^2) = Tr(\alpha) + Tr(\lambda_1^2 \lambda_2^2) = Tr(\alpha) + Tr(\lambda_1 \lambda_2) = 1,$$

we see again that this equation has no solutions and therefore we find a contradiction. \hfill $\square$
Corollary 2.47. If \( q \) is an even non-square, then
\[
\alpha(ER_q) \geq \frac{1}{\sqrt{2}}q^{3/2} - q + \frac{1}{\sqrt{2}}\sqrt{q}.
\]

II. \( q \) an odd square and \( \sqrt{q} \equiv 3 \pmod{4} \)

It is a classical result by Segre that when \( q \) is odd, the absolute points of a orthogonal polarity are a non-degenerate conic. Therefore, we might use the term conic of absolute points, when we talk about the set of absolute points. Moreover, every non-degenerate conic defines a polarity. We again start by fixing a coordinate system. Let \( C \) be the conic of absolute points, defined by the equation
\[
X_1^2 - X_0X_2 = 0,
\]
and \( \perp \) the orthogonal polarity of \( P\Gamma(2, q) \) defined by this conic. Then the conic \( C \) consists of the points \( \{(t, t^2) \mid t \in GF(q)\} \cup \{(0, 0, 1)\} \) and for a point \( x = (x_0, x_1, x_2) \in P\Gamma(2, q) \) the line \( x^\perp \) is defined by the equation \( x_2X_0 - 2x_1X_1 + x_0X_2 = 0 \). The isomorphism between \( P\Gamma O(3, q) \) and \( P\Gamma L(2, q) \) is now given by
\[
\begin{pmatrix}
a^2 & 2ac & c^2 \\
ab & ad + bc & cd \\
b^2 & 2bd & d^2
\end{pmatrix} \leftrightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},
\]
where \( a, b, c, d \in GF(q) \) and \( ad - bc \neq 0 \). When \( q \) is square, we can consider a Baer subplane \( B \) of \( P\Gamma(2, q) \). This subplane is isomorphic to \( P\Gamma(2, \sqrt{q}) \) and \( C \cap B \) is a conic \( c \) of \( P\Gamma(2, \sqrt{q}) \). We can consider the subgroup \( H \) of \( P\Gamma L(3, q) \) that leaves this small conic invariant. This subgroup is clearly isomorphic to \( P\Gamma L(2, \sqrt{q}) \). This is the group of which we will consider a fitting orbit. In order to do so, we need to know how this group acts on \( P\Gamma(2, q) \). We already know that in the Baer subplane, it has three orbits: the points on \( c \), the external points and the internal points (with respect to the conic \( c \)). When looking at the whole plane, the situation is a bit more difficult. Denote the external points and internal points of \( P\Gamma(2, q) \) (with respect to \( C \)) again as \( Ext(q) \) and \( Int(q) \).

Lemma 2.48. The group \( H = P\Gamma L(2, \sqrt{q}) \) has the following orbits on \( P\Gamma(2, q) \setminus B \):

- one orbit of size \( q - \sqrt{q} \) of points of \( C \setminus c \),
- one orbit of size \( q^{3/2} - \sqrt{q} \) of points of \( Ext(q) \),
- \( \sqrt{q} - 2 \) orbits of size \( (q^{3/2} - \sqrt{q})/2 \) of points of \( Ext(q) \),
- \( \sqrt{q} \) orbits of size \( (q^{3/2} - \sqrt{q})/2 \) of points of \( Int(q) \).

In order to complete the calculations, we need the following lemmas, which can be found in [42] or which are standard finite field-theoretic lemmas. Denote by \( \Box_q \) the subgroup of squares of \( GF(q) \setminus \{0\} \) and \( \Box_{\sqrt{q}} \) in a similar way. Remark that \( GF(\sqrt{q}) \subseteq \Box_q \) and \( -1 \notin \Box_{\sqrt{q}} \) (as \( \sqrt{q} \equiv 3 \pmod{4} \) but by the previous remark, \( -1 \in \Box_q \)).

Lemma 2.49. A point \( x = (x_0, x_1, x_2) \in P\Gamma(2, q) \) is external to \( C \) if and only if \( x_1^2 - x_0x_2 \in \Box_q \setminus \{0\} \).

Lemma 2.50. Let \( a \in GF(q) \), then \( a\sqrt{q+1} \in \Box_{\sqrt{q}} \) if and only if \( a \in \Box_q \).

Theorem 2.51. If \( q \) is an odd square and \( \sqrt{q} \equiv 3 \pmod{4} \), then there exists a set of points \( I \) of size \( (q^{3/2} - \sqrt{q})/2 + q + 1 \), such that for every \( x \in I \) we have \( x^\perp \cap I = 0 \).
Proof. Let \( w \in GF(q) \) be a non-square, i.e., \( w \notin \square_q \), and consider the point \( x = (1, 0, w) \). By Lemma 2.49 we see that \( x \) is internal as \( -w \notin \square_q \). Then \( x^\perp \) is defined by \( wX_0 + X_2 = 0 \). Now let \( I \) be the following set: \( I = xH \cup C \). Then, by Lemma 2.48 \(| I | = q^{3/2} - \sqrt{q} + q + 1 \). As the absolute points are an independent set, and no absolute point is adjacent to an internal point, it suffices to prove that \( x^H \) is an independent set in \( ER_q \). We will show that \(| x^\perp \cap x^H | = 0 \), which proves the theorem, applying the same argument as in equation (2.4).

So assume the intersection is not empty. Using the explicit form of \( H \), we see that

\[
x^H = \{(a^2 + c^2 w, ab + cdw, b^2 + d^2 w) \mid a, b, c, d \in GF(\sqrt{q}), ad - bc \neq 0 \}.
\]

Substituting this in the equation of \( x^\perp \), we see that

\[
c^2 w^2 + (a^2 + d^2)w + b^2 = 0.
\]

Suppose that \( c = 0 \), then if \( a^2 + d^2 \neq 0 \) we would have that \( w = b^2/(a^2 + d^2) \in GF(\sqrt{q}) \), which is a contradiction. So \( a^2 + d^2 = 0 \), which would imply that \( a = d = 0 \), as \(-1\) is not a square in \( GF(\sqrt{q}) \) and so \( ad - bc = 0 \) which is again a contradiction. Therefore we see that \( c \neq 0 \). However, if \( w \) is a solution of the quadratic equation, then so is \( w\sqrt{q} \) as can be seen by applying the Frobenius automorphism \( x \mapsto x\sqrt{q} \) to the equation. Then \( w\sqrt{q}+1 = b^2/c^2 \notin \square_q \), which means by Lemma 2.50 that \( w \in \square_q \), which is again a contradiction. We conclude that in all cases we find contradictions. Therefore our initial assumption of the non-empty intersection is false and we find that \(| x^\perp \cap I | = 0 \). \( \square \)

Corollary 2.52. If \( q \) is an odd square and \( \sqrt{q} \equiv 3 \pmod{4} \), then

\[
\alpha(ER_q) \geq \frac{q^{3/2} - \sqrt{q}}{2} + q + 1.
\]

III. \( q \) an odd square and \( \sqrt{q} \equiv 1 \pmod{4} \)

We use the notation from the previous part. The conic of absolute points is \( C : X_1^2 - X_0X_2 = 0 \). A point \( x = (x_0, x_1, x_2) \in PG(2, q) \) has as polar image \( x^\perp : x_2X_0 - 2x_1X_1 + x_0X_2 = 0 \). Consider the subgroup \( K \) of PGL(2, q) given by the matrices

\[
\begin{pmatrix}
a^2 & 2ac & c^3 \\
0 & a & c \\
0 & 0 & 1
\end{pmatrix},
\]

where \( a, c \in GF(q) \) and \( a\sqrt{q}+1 = 1 \). It is a standard fact that the equation \( x\sqrt{q}+1 = 1 \) has \( \sqrt{q} \) solutions in \( GF(q) \) and so this subgroup has size \( q(\sqrt{q} + 1) \). It can be shown that this subgroup permutes the \( q(q - 1)/2 \) internal points in \( \sqrt{q} - 1 \) orbits, each of size \( q(\sqrt{q}+1)/2 \). Again, we need a small lemma for the computations in our main result.

Lemma 2.53. If \( a \in GF(q) \) such that \( a\sqrt{q}+1 = 1 \), then \( a^2 + 1 \in \square_q \).

Proof. Let \( i \) be the root of a non-square \( s \) in \( GF(\sqrt{q}) \). Hence, \( i\sqrt{q} = -i \), \( i^2 = s \) and \( GF(q) = \{ a + bi \mid a, b \in GF(\sqrt{q}) \} \). Suppose we have \( a \in GF(q) \) such that \( a\sqrt{q}+1 = 1 \), then we can write \( a = b + ci \) where \( b, c \in GF(\sqrt{q}) \). As \( a\sqrt{q} = b - ci \), we see that \( a\sqrt{q}+1 = b^2 - sc^2 = 1 \) and hence \( sc^2 + 1 = b^2 \). Therefore \( a^2 + 1 = b^2 + 2bc + sc^2 + 1 = 2b(b + ci) \) and \( (a^2 + 1)\sqrt{q}+1 = 4b^2 \in \square_q \). By Lemma 2.50 it follows that \( a^2 + 1 \in \square_q \). \( \square \)

Theorem 2.54. If \( q \) is an odd square and \( \sqrt{q} \equiv 1 \pmod{4} \), then there exists a set of points \( I \) of size \( (q^{3/2}/2 + q + 1) \) such that for every \( x \in I \) we have \(| x^\perp \cap I | = 0 \).

The proof is almost exactly the same as the one from Theorem 2.51.
The quadrilateral

Proof. Let \( w \in GF(q) \) again be a non-square, i.e., \( w \notin \mathbb{F}_q \), and consider the point \( x = (1, 0, w) \). By Lemma 2.49 we see that \( x \) is internal as \( -w \notin \mathbb{F}_q \). Then \( x^\perp \) is defined by \( wX_0 + X_2 = 0 \). Now let \( I \) be the following set: \( I = x^K \cup C \). Then \( |I| = 23^{1/2} + q + 1 \). As the absolute points are an independent set, and no absolute point is adjacent to an internal point, it suffices to prove that \( x^K \) is an independent set in \( ER_q \). We will show that \( |x^\perp \cap x^K| = 0 \), which proves the theorem, applying the same argument as in equation (2.4).

So assume the intersection is not empty. Using the explicit form of \( K \), we see that

\[
x^K = \left\{ (a^2 + c^2 w, cw, w) \mid a, c \in GF(\sqrt{q}), a\sqrt{q} + 1 = 1 \right\}.
\]

Substituting this in the equation of \( x^\perp \), we see that, using \( w \neq 0 \),

\[
c^2 w + a^2 + 1 = 0.
\]

Suppose that \( c = 0 \) then we have

\[
\begin{align*}
a\sqrt{q} + 1 &= 1 \\
a^2 &= -1,
\end{align*}
\]

which is impossible as this means that \( 4 \) divides \( (\sqrt{q} + 1) \equiv 2 \) (mod 4). Therefore we see that \( c \neq 0 \). Then \( w = -(a^2 + 1)/c^2 \in \mathbb{F}_q \) by Lemma 2.53 which is again a contradiction. We conclude that \( |x^\perp \cap I| = 0 \).

\( \square \)

Corollary 2.55. If \( q \) is an odd square and \( \sqrt{q} \equiv 1 \) (mod 4), then

\[
\alpha(ER_q) \geq \frac{q^{3/2} + q}{2} + q + 1.
\]
Chapter 3

Bipartite complete graphs

There are two natural ways to generalize the results from previous chapter. As $C_4 = K_{2,2}$, we can either choose to consider other complete bipartite graphs $K_{s,t}$ or other even cycles $C_{2k}$ as the forbidden subgraph. Their generalisations are of a completely different nature: one class is the densest among bipartite graphs, while the other is a class of very sparse graphs. It turns out that the first generalisation, to complete bipartite graphs, is the most fruitful one. One of the reasons is that forbidding $K_{s,t}$ implies that every $s$ vertices have at most $t-1$ neighbours, which is a local property: it can be checked using only the neighbourhoods of vertices. On the other hand, checking if a graph does not contain a cycle $C_{2k}$ is not so obvious, especially as $k$ gets larger.

We will start this chapter of with an introduction to designs. These objects come quite naturally when considering $K_{s,t}$-free graphs, exactly like projective planes appeared naturally when considering $K_{2,2}$-free graphs. This is no coincidence, as projective planes are a special case of designs.

3.0 Foundations: design theory

3.0.1 Basics

This part is largely based on [21].

Let $v, k, t, \lambda$ be non-negative integers such that $v > k \geq t \geq 2$.

**Definition 3.1.** A $t$-$(v, k, \lambda)$ design, or $t$-design when the other parameters are clear from the context, is a combinatorial structure consisting of a pair of non-empty sets $(\mathcal{P}, \mathcal{B})$. The elements of $\mathcal{P}$ are the **points** and those of $\mathcal{B}$ are the **blocks** of the design. They satisfy three axioms:

1. $|\mathcal{P}| = v$;
2. every block contains exactly $k$ points, i.e., $\mathcal{B} \subseteq \binom{\mathcal{P}}{k}$;
3. every $t$ distinct points are contained in exactly $\lambda$ blocks.

We often say that a point is **incident** with a block, the block **goes through** the point or the block is **incident** with the point, if the block contains the point. In contrast with graphs, we do not denote the point and block set of a design $D$ by $\mathcal{P}(D)$ and $\mathcal{B}(D)$ respectively, but by $\mathcal{P}$ and $\mathcal{B}$, as we normally only consider one design at a time. If confusion might arise however, we will use this notation to make the distinction.

We are already familiar with a certain type of designs as the following theorem shows. It also shows a relation between the topics of design theory and finite geometry.

1. These terms are the same as for point-line geometries. One could actually define designs as certain point-line geometries, but we prefer not to.
Theorem 3.2. Let \( n \geq 2 \). The \( 2-(n^2 + n + 1, n + 1, 1) \) designs are exactly the projective planes of order \( n \), where the blocks are the lines and vice-versa.

Proof. Assume we have a \( 2-(n^2 + n + 1, n + 1, 1) \) design \( D \), then we need to show that the points and blocks form a linear space that satisfies axioms (PP1) and (PP2). It is clear that it is a linear space, by axiom (D3).

(PP1) as through every two points, there is a unique line, and every line contains \( n + 1 \) points, we find: 

\[
\binom{n^2 + n}{n} = n + 1 \text{ lines.}
\]

Now take two lines \( L_1, L_2 \) and \( x \in L_2 \setminus L_1 \). Every point on \( L_1 \) together with \( x \) determines a unique line, giving rise to \( n + 1 \) lines through \( x \). As these are all the lines through \( x \), \( L_2 \) should be one of them and hence \( L_1 \) and \( L_2 \) should intersect in a point.

(PP2) Take two lines \( L_1, L_2 \), then we can take their point of intersection \( x \) and one point on each, distinct from \( x \), say \( x_1, x_2 \). Then we have \( n^2 + n + 1 - 2(n + 1) + 1 = n^2 - n > 0 \) points left, not on any of those lines, take one of them. This gives rise to four points, of which no three collinear.

We conclude that \( D \) is a projective plane. As we have \( n^2 + n + 1 \) points, \( n + 1 \) points on every line and every line contains \( n + 1 \) points, as computed in the proof of (PP1), we see that it is of order \( n \).

Now suppose we have a projective plane of order \( n \), then (D1), (D2) and (D3) can be easily checked. \( \square \)

Now back to the more general definition, we can immediately prove the following theorem from the axioms. We recall the convention that parameters reflect non-negative integers unless otherwise stated.

Theorem 3.3. Any \( t \)-design is also an \( s \)-design for any \( s \leq t \).

Proof. Suppose we have a \( t-(v, k, \lambda) \) design \( D \), then we need to prove that for any \( s \leq t \) there is a constant number of blocks containing any set of \( s \) points. So fix \( s \leq t \) and let \( X \subseteq \mathcal{P} \) be a set of \( s \) points. By double counting pairs \( (Y, B) \), where \( Y \) is a set of \( t \) points containing \( X \) and \( B \in \mathcal{B}, Y \subseteq B \) we find:

\[
\binom{v-s}{t-s} \lambda = \lambda_X \binom{k-s}{t-s},
\]

where \( \lambda_X \) is the number of blocks containing \( X \). Clearly this is independent of the chosen set \( X \), which is what we needed to prove. \( \square \)

There are two particular instances in which we can use this theorem. First, we can compute the number of blocks, denoted by \( b \), by putting \( s = 0 \):

\[
b = \lambda \binom{v}{k}. 
\]

Secondly, the number of blocks through any point is constant; this constant is commonly denoted by \( r \) and can be computed by putting \( s = 1 \). Double counting incident point-block pairs we find

\[
bk = vr.
\]

We can derive some more inequalities between these parameters. Let \( t \geq 2 \) and \( D \) be a \( t-(v, k, \lambda) \)-design. Take a block \( B \) and a point \( x \in B \). Take a set \( X \) of \( t \) points such that \( x \in X \), \( X \not\subseteq B \). Through these \( t \) points, there are \( \lambda \) blocks by (D3), all different from \( B \). These \( \lambda \) blocks, together with \( B \) give us \( \lambda + 1 \) blocks through \( x \) and hence \( r \geq \lambda + 1 \).

We can also consider the incidence matrix \( M \) of \( D \), where the rows are indexed by the points and the columns by the blocks. Hence \( M_{ij} = 1 \), when the point \( p_i \) is contained in the block \( M_j \) and zero
otherwise. Then we can compute \( MM^T \). This \( v \times v \)-matrix is independent of the chosen ordering of the points and blocks and has

\[
(MM^T)_{ij} = \sum_{k=1}^{b} M_{ik} M_{jk} = \begin{cases} r & \text{if } i = j, \\ \lambda & \text{if } i \neq j. \end{cases}
\]

One can compute the determinant of this matrix and find \( \det(MM^T) = (r + (v - 1)\lambda)(r - \lambda)^{v-1} \). As \( r > \lambda \), it follows that this matrix has rank \( v \) and hence \( v = \text{rk}(MM^T) = \text{rk}(M) \leq b \). Therefore \( v \leq b \), which is called Fisher's inequality.

Finally, we discuss a way to find new designs from existing designs, we will use this in a proof later on. Let \( t \geq 3 \), \( k \geq 3 \) and \( D \) be a \( t-(v, k, \lambda) \)-design. Then, fixing a point \( x \) of \( D \), we can construct a new design: the points are all points of \( D \) except \( x \), and the blocks are the blocks through \( x \) (with \( x \) removed then). Doing this, one can check that we have constructed a \((t-1)-(v-1, k-1, \lambda)\)-design, it is commonly called the derived design \( D_x \).

### 3.0.2 Symmetric designs

Certain designs achieve equality in Fisher’s inequality, the so-called symmetric designs, having \( v = b \). This might be a misleading term as it does not mean that the incidence matrix \( M \) is symmetric. It is for this reason that many authors prefer to call them square designs, as \( k \) in fact a square matrix. We will however stick to the terminology of symmetric designs, as this is the way they occur most frequently in the literature. Remark that combining the equalities \( bk = vr \) (derived before) and \( b = v \) (by definition of symmetric design), we see that also \( k = r \) holds true in a symmetric design. We have already seen an example of a symmetric design: that of a projective plane of order \( n \), which is a design as we saw in Theorem 3.2. It is a symmetric design as it has \( n^2 + n + 1 \) points and lines.

First of all, we have the following important theorem that limits the options for the parameters of a symmetric design.

**Theorem 3.4.** Let \( t \geq 3 \), then a symmetric \( t-(v, k, \lambda) \)-design has \( k = v - 1 \).

When \( k = v - 1 \), the design is often called a trivial design as it has no interesting combinatorial properties. For example, every block contains all but one point so through \( t \) points there are \( b-t = v-t \) blocks. It follows that \( \lambda = v-t \). A trivial design therefore has parameters \( t-(v, v-1, v-t) \).

**Proof.** Let \( D \) be a symmetric \( t-(v, k, \lambda) \)-design with \( t \geq 3 \). Then we can take a point \( x \) of the design and consider \( D_x \). As \( t \geq 3 \), we have \( t-1 \geq 2 \) and hence Fisher’s inequality still holds. Now the number of blocks in the derived design is the number of blocks through \( x \), which is \( r \). Hence \( v-1 \leq r \). As \( D \) is symmetric, \( k = r \) and hence \( v-1 \leq k \). However, by the definition of design, we have \( v > k \) and therefore \( v-1 = k \).

We conclude that the only symmetric designs with \( t \geq 3 \) are \( t-(v, v-1, v-t) \)-designs, which have little combinatorial interest as mentioned before, in particular because \( \lambda \) cannot be kept constant when varying \( v \). This means, when trying to construct graphs from these designs, we cannot construct more than one for any given \( t \geq 3 \) and \( \lambda \geq 1 \). Consequently, the interesting symmetric designs have \( t = 2 \). If \( \lambda = 1 \), we have \( v-1 = k(k-1) \). This can be seen by fixing a point and double counting point-block pairs, such that both points are contained in the block, using that \( r = k \). Then, introducing the parameter \( n = k - 1 \), which is called the order of the design, we see that a symmetric design with \( \lambda = 1 \) is a \( 2-(n^2 + n + 1, n + 1, 1) \)-design, which are exactly the projective planes as we saw in Theorem 3.2. Because of this correspondence, we know that there are infinitely many such designs. On the other hand, when \( \lambda = 2 \), there are only 7 known values of \( v \) for which these symmetric designs,
the so-called biplanes, exist. Even more, a folklore conjecture states that for every $\lambda > 1$, there exist only finitely many symmetric designs. We will however in the following sections continue to work with these objects, disregarding their possible non-existence. The reason is that they possess very good combinatorial properties, which might unfortunately, as the conjecture tells us, be too good to be true.

We show the reason why exactly symmetric designs are of interest to us. It is captured in the following theorem.

**Theorem 3.5.** Let $D$ be a $2-(v, k, \lambda)$-design, then $D$ is symmetric if and only if every two blocks intersect in exactly $\lambda$ points.

Recall that in a $2-(v, k, \lambda)$-design, two points are contained in exactly $\lambda$ blocks. This shows that in a symmetric design we also have the dual property and that this property defines symmetric designs.

**Proof.** Suppose $D$ is symmetric and let $M$ be the incidence matrix of $D$, with respect to a certain ordering of the points and blocks (again points as rows and columns as blocks). We can use this matrix to find in how many points two distinct blocks intersect, namely by computing $M^T M$. We already know that the diagonal elements equal $k = r = \det(M) \neq 0$ and so $M$ is invertible. Using this, we can compute $M^T M$:

$$M^T M = M^{-1}(MM^T)M = M^{-1}((r - \lambda)I + \lambda J)M = (r - \lambda)I + M^{-1}\lambda k J = (r - \lambda)I + M^{-1}\lambda r J = (r - \lambda)I + \lambda J,$$

where we used that $JM = kJ$, $MJ = rJ$ and $k = r$. We can see the off-diagonal elements are also $\lambda$ in every position and hence every two blocks intersect in exactly $\lambda$ points. 

Now suppose every two blocks intersect in exactly $\lambda$ points, then we can switch the roles of points and blocks and find a $2-(b, r, \lambda)$-design. By Fisher’s inequality, applied to this design, we see that $b \leq v$. However, we can also apply the inequality to the original design $D$, so $v \leq b$ and hence $v = b$. 

Take a step back and look at both of these properties: through two points go exactly $\lambda$ blocks and two blocks have exactly $\lambda$ points in common. This might already give an idea regarding the construction of $K_{2, \lambda + 1}$-free graphs. We will explore this connection deeper in Section 3.3. Before that, we review the main theorems in this subfield of Turán numbers.

### 3.1 General results

The first, most important result is one due to Kővari, Sós and Turán [49]. Before we can prove this, we need a lemma. It is a special case of Jensen’s inequality and a generalisation of Lemma 1.6. For the reader that is not familiar with this inequality, we prove the special case here.

**Lemma 3.6.** Let $k, n, d_1, \ldots, d_n$ be non-negative integers, then

$$\left(\sum_{i=1}^{n} d_i\right)^k \leq n^{k-1} \left(\sum_{i=1}^{n} d_i^k\right).$$

To prove this, we need to use the concept of convex functions, which is what Jensen’s inequality is about. A function $f : \mathbb{R} \to \mathbb{R}$ is convex in an interval $I \subseteq \mathbb{R}$ if and only if for every $x, y \in I$ and $\alpha \in [0, 1]$, we have $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y).$ One can show (out of the scope of this thesis) that a twice differentiable function is convex on an interval if and only if its second derivative
is non-negative there. In our case the function will be \( f(x) = x^k \), where \( k \) is a non-negative integer. By the criterion we see immediately that this function is convex on \([0, \infty)\). The inequality that defines convex functions can also be seen graphically. Loosely speaking, they slope upwards on the interval that they are convex on. We show this behaviour for the function \( f(x) = x^k \) in Figure 3.1.

**Figure 3.1:** The function \( f(x) = x^k, k > 1 \), is convex on the interval \([0, +\infty)\).

**Proof of Lemma 3.6** We use convexity to each time ‘cut off’ one term. First rewrite the inequality we need to prove as

\[
\left( \sum_{i=1}^{n} d_i \right)^k \leq \frac{1}{n} \left( \sum_{i=1}^{n} d_i^k \right).
\]

Then as \( \frac{n-1}{n} + \frac{1}{n} = 1 \),

\[
\left( \sum_{i=1}^{n} d_i \right)^k = \left( \frac{n-1}{n} \sum_{i=1}^{n-1} d_i + \frac{d_n}{n} \right)^k \leq \frac{n-1}{n} \left( \sum_{i=1}^{n-1} d_i \right)^k + \frac{d_n^k}{n}.
\]

by convexity. We can continue in this fashion to find the inequality we want.

**Theorem 3.7.** For all \( s, t \geq 1 \), we have

\[
\text{ex}(n, K_{s,t}) < \frac{1}{2} (t-1)^{1/s} n^{2-(1/s)} + \frac{s}{2} n.
\]

**Remark 3.8.** Originally, we wanted to use the proof in Ball’s book [8], but it contains an imprecision which seems not easy to solve. To be more precise: the application of Lemma 5.8 is incorrect in line 10-11 of the proof. The argument can probably be kept after some refinement, but we opted to use the original proof due to Kővari, Sós and Turán instead.
Proof. Denote the upper bound in the statement by $e$. Let $G$ be a $K_{s,t}$-free graph on $n$ vertices having $e(G) = e$. We prove the statement by double counting the number of stars $S_n = K_{s,1}$ in $G$, denote this number by $N$. On the one hand, as $G$ does not contain $K_{s,t}$, we have

$$N \leq \binom{n}{s} (t - 1). \tag{3.1}$$

Let $V(G) = \{v_1, \ldots, v_n\}$ and $d_i = d(v_i)$, arranged such that for $i \leq m$, we have $d_i \geq s$ and for $i > m$ that $d_i < s$, for some $m$ between 1 and $n$. If $m$ were zero, then $e(G) < \frac{1}{2}ns$ which is smaller than $e$ for $n$ large enough.

$$N = \sum_{i=1}^{n} \left( \binom{d_i}{s} \right) = \sum_{i=1}^{m} \left( \binom{d_i}{s} \right) > \frac{1}{s!} \sum_{i=1}^{m} (d_i - s)^s.$$

Using the lemma we find the following series of inequalities:

$$N > \frac{1}{s!} n^{1-s} \left( \sum_{i=1}^{m} (d_i - s) \right)^s$$

$$> \frac{1}{s!} n^{1-s} \left( \sum_{i=1}^{m} d_i - ms \right) - (n - m)s + d_{m+1} + \cdots + d_n \right)^s$$

$$> \frac{1}{s!} n^{1-s} (2e(G) - ns)^s$$

$$> \frac{1}{s!} n^{1-s} ((t - 1)n^{2s-1})$$

$$> \frac{1}{s!} (t - 1)n^s$$

$$> (t - 1) \binom{n}{s}, \tag{3.2}$$

using in the second line that $d_i < s$ when $i > m$. Now comparing (3.1) with (3.2), we find a contradiction. \qed

Remark that the roles of $s$ and $t$ can be switched, therefore, the above bound is most effective when $s \leq t$. The most important part of the bound is the main term. It gives in many cases the correct magnitude for $\text{ex}(n, K_{s,t})$ as we will see later. For example when $s = 1$, then $K_{1,t} = S_{t+1}$, we saw in Chapter 1 that

$$\text{ex}(n, S_{t+1}) = \left\lfloor \frac{1}{2} (t - 1)n \right\rfloor,$$

which is indeed of the same magnitude and has the same constant as one would obtain from Theorem 3.7. In Chapter 2, we found that $\text{ex}(n, K_{2,2}) = \Theta(n^{3/2})$ and Theorem 3.7 also gave the correct constant. This will not always happen as we will see in the example of $\text{ex}(n, K_{3,3})$ later on, where the constant will be $1/2$ as opposed to $\sqrt{2}/2$, as predicted by Theorem 3.7.

Lastly, it has not yet been determined whether $\text{ex}(n, K_{s,t}) = \Theta(n^{2 - (1/s)})$ for all $s, t$ such that $s \leq t$. It is conjectured that this is correct for all $s \leq t$. We will discuss some cases where this is indeed the correct magnitude.

**Conjecture 3.9.** If $s \leq t$ then $\text{ex}(n, K_{s,t}) = \Theta(n^{2 - (1/s)})$.

Now we show the only known general lower bound. Recall that to prove a lower bound for $\text{ex}(n, K_{s,t})$, we need to construct a $K_{s,t}$-free graph for every $n$. This can be done by the probabilistic method, for which we assume some knowledge of probability theory. For someone who is unfamiliar with this technique, this method might seem odd. We will not actually construct the graph, but merely show
its existence. The technique goes as follows: we ‘randomly’ construct a $K_{s,t}$-free graph and compute its expected number of edges. If this expected number is larger than a constant $C$, then there should be a graph that has at least this number of edges. We just need to explain how we can randomly construct a graph. This is done in what is called the $G(n, p)$ model. In this model, we construct a graph $G$ starting from $n$ isolated vertices and adding every possible edge independently with probability $p$, $0 < p < 1$. The fact that the edges occur independently is very important when we want to count things, as we will see in the following proof. The probability $p$ can be dependent or independent from $n$, depending on the goal in mind, but in most existence proofs, $p$ will be chosen appropriately in function of $n$.

**Theorem 3.10.** For all $s, t \geq 2$ we have

$$\text{ex}(n, K_{s,t}) > cn^{2-(s+t-2)/(st-1)},$$

for some constant $c$ depending only on $s$ and $t$.

When comparing this result to Theorem 3.7 and Conjecture 3.9, it is clear that this bound is always smaller in magnitude, therefore we do not care about the explicit value of $c$. For example, for $s = t = 2$, we find a lower bound with main term $n^{4/3}$ which is of smaller magnitude than $n^{3/2}$, obtained by the polarity graph.

**Proof.** We work in the $G(n, p)$ model, constructing the random graph $G$. Let $X$ be the number of edges in $G$. This is a random variable and hence not a number. We can however compute the expected number of edges:

$$\mathbb{E}(X) = \binom{n}{2} p > cn^2 p,$$

for any $c < \frac{1}{2}$ if $n$ is large enough. Similarly, if $Y$ is the number of copies of $K_{s,t}$ in $G$, then

$$\mathbb{E}(Y) = \frac{n}{s} \binom{n-s}{t} p^{st} < c'n^{s+t}p^{st},$$

where $c' = 1/(s!t!)$. We can make $G$ $K_{s,t}$-free by removing an edge from every copy of $K_{s,t}$. The remaining number of edges is then $X - Y$, so by linearity of expectations, the expected number of edges remaining is

$$\mathbb{E}(X - Y) = \mathbb{E}(X) - \mathbb{E}(Y) > cn^2 p - c'n^{s+t}p^{st}.$$ 

Choosing $p = \left(\frac{c}{2cn^2}\right)^{1/(st-1)} n^{-(s+t-2)/(st-1)}$, one can compute

$$\mathbb{E}(Y - X) > c''n^{2-(s+t-2)/(st-1)},$$

for some $c''$ which can be made explicit. Hence, there exists a $K_{s,t}$-free graph on at least the above number of edges, as we wanted. 

### 3.2 Finding good lower bounds

#### 3.2.1 The norm graph

In this section, we construct some $K_{s,t}$-free graphs for several values of $s$ and $t$, still assuming that $s \leq t$. First, we look at off-diagonal Turán numbers. By this we mean $\text{ex}(n, K_{s,t})$ when $t$ is much larger then $s$. This was first done in 1996 by Kollár, Rónyai and Szabó for $t \geq s! + 1$ when they constructed the norm graph [47]. Later, this graph was slightly changed and the result got improved by Alon, Rónyai and Szabó to $t \geq (s-1)!+1$ [5]. It is the only result showing $\text{ex}(n, K_{s,t}) = \Theta(n^{2-(1/s)})$ for infinitely many $s$. 

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Theorem 3.11. Let $s \geq 2$, then there exists a constant $c$ such that

$$\text{ex}(n, K_{s,(s-1)!+1}) \geq \frac{1}{2} n^{2-(1/s)} + O(n^{2-(1/s)-c}).$$

Their construction involves finite field theory and some results of algebraic geometry. We can construct the graph, but not show that it is $K_{s,(s-1)!+1}$-free.

Construction 3.12. Take the field $GF(q^{s-1})$ and define the norm of an element to be $N(x) = x^{1+q+q^2+\cdots+q^{s-2}}$. Then define the norm graph $H(q,s)$ in the following way: the vertices are the elements of $GF(q^{s-1}) \times GF(q)^*\setminus \{0\}$ with multiplication. Two vertices $(x,a)$ and $(y,b)$ are adjacent if and only if $N(x+y) = ab$. We see that $v(G) = q^s - q^{s-1}$ and one can show that this graph is regular of degree $q^{s-1} - 1$.

Remark that as $K_{s,t} \subseteq K_{s,t'}$ for all $t' \geq t$, we find that

$$\text{ex}(n, K_{s,t}) = \Theta(n^{2-(1/s)}),$$

for all $t \geq (s-1)! + 1$.

Last year, Ball and Pepe translated the norm graph into finite geometry, in order to prove the following result [9].

Theorem 3.13. The graph $H(q,s)$ is $K_{s+1,(s-1)!-1}$-free.

The finite geometry they use is still out of the scope of this thesis, but using their geometric setting we can later on consider a graph that is a slight adaptation of $H(q,3)$ and shows $\text{ex}(n, K_{3,3}) = \Theta(n^{5/3})$. The preceding result does not give a better bound than the probabilistic bound we found earlier when $t \geq 5$. However, it is a deterministic proof, which is generally preferable to non-constructive proofs. For $s = 4$, it gives the best known lower bound for $\text{ex}(n, K_{5,5})$. Finally, for $s = 2$, the construction gives a subgraph of $ER_q$.

3.2.2 Excluding $K_{3,3}$

Already in 1966, in the same paper where Brown proved that $\text{ex}(n, K_{2,2}) = \Theta(n^{3/2})$ using the polarity graph, he gave a construction showing that $\text{ex}(n, K_{3,3}) = \Theta(n^{5/3})$. His construction is based on the fact that three unit balls in $\mathbb{R}^3$ have at most two common points of intersection. He translated this idea to finite geometry and thus found a finite $K_{3,3}$-free graph on $n$ vertices and approximately $n^{5/3}$ edges. This construction cannot be extended to higher dimension as we can find two orthogonal unit spheres in dimension 4 already, i.e. every point from one sphere lies on distance one from the other. When comparing this construction to the norm graph of before, the similarity is not hard to find: distances already relate to norms in Euclidean space, so the ideas behind the construction are not completely unrelated. We will not show this construction, but give the one due to Ball and Pepe [9] we discussed before. To start off, we need the concept of an ovoid.

Definition 3.14. An ovoid of $PG(3,q)$ is a set $\mathcal{O}$ of points, no three collinear, such that to every point $x \in \mathcal{O}$, there is a unique plane $\pi_x$ associated such that for any line $L$, $L \cap \mathcal{O} = \{x\}$ if and only if $L \subseteq \pi_x$.

We have to make a quick remark regarding an imprecision in Ball’s book. Those not familiar with ovals and hyperovals, can skip it and continue to the proof without any problem.

2. Remark that this norm does not coincide with the one we defined in Chapter 2. This a relative field norm, as opposed to the absolute field norm we defined in Chapter 2.
Remark 3.15. In Ball’s book, p.48, the definition of ovoid contains a slight imprecision when considering the proofs that follow it. It is defined as a subset of points $PG(k, q)$, no three of which collinear with the additional property that ‘the tangents to $O$ containing $x$ are all lines incident with $x$ in some hyperplane $H$’. Although this definition seems more or less the same as ours, this one does not imply the existence of tangent lines. It states that if there are tangent lines, they lie in a hyperplane, but it might happen that there are none. For example, a hyperoval satisfies the definition but contradicts the theorems that follow the definition: he proves that an ovoid in $PG(2, q)$ has $q + 1$ points, while a hyperoval has $q + 2$. Another example occurs when one takes $O = PG(n, 2) \setminus H$ in $PG(n, 2)$, where $H$ is a hyperplane of $PG(n, 2)$: this also satisfies the definition as no three points are collinear and the requirement in regards to tangent lines is void, but the number of points is larger than he proves later on.

Let $O$ be an ovoid in $PG(3, q)$. If $|L \cap O| = 1$ then $L$ is a tangent line of $O$. All the tangent lines through a point lie in a plane by definition, so every other line through that point, not in the plane, has to intersect $O$ in exactly two points. This means that $|O| = q^2 + 1$. Moreover, let $\pi$ be a plane in $PG(3, q)$ and assume it contains a point $x \in O$. As two planes intersect in at least a line in $PG(3, q)$, $\pi$ contains at least one tangent line through $x$. Considering the other lines through $x$ in $\pi$, we can conclude that a plane has at most $q + 1$ points in common with $O$. As a final remark, ovoids do exist and can be constructed for all prime powers $q$. The graph construction in the proof below can thus be executed for these $q$.

Theorem 3.16. For all $\epsilon > 0$ there exists $n_0$, such that for all $n \geq n_0$ we have,

$$\text{ex}(n, K_{3,3}) \geq \frac{1}{2}(1 - \epsilon)n^{5/3}.$$ 

Proof. Consider $PG(4, q)$, and let $\beta$ be an orthogonal polarity of this space. Let $z \in PG(4, q)$ be a point such that $z \notin z^\perp$ and take an ovoid $O$ in $z^\perp$ (which is indeed a three-dimensional space). We construct a graph $G$ by taking as vertex set all points on lines so where $o \in O$, $z$ and $o$ excluded. Therefore, we have $q^2 + 1$ lines to consider, each containing $q + 1 - 2 = q - 1$ points of $V(G)$. This gives a total of $n = (q^2 + 1)(q - 1) \approx q^3$ vertices. Two vertices $x, y \in V(G)$ are adjacent if and only if $x \in y^\perp$ or equivalently $y \in x^\perp$.

Each vertex $x$ has degree at least $q^2 + 1 - (q + 1) - 1$: as the three-dimensional space $x^\perp$ does not contain $z$ (because $x \notin z^\perp$), it intersects the $q^2 + 1$ lines through $z$ in $q^2 + 1$ distinct points. Moreover, at most $q + 1$ of these can be points of $O$ as the intersection of $x^\perp$ and $z^\perp$ is at most a plane, which contains at most $q + 1$ points of an ovoid, as we saw above. These $q + 1$ points are not vertices of the graph and hence at least $q^2 + 1 - (q + 1)$ points remain as vertices of the graph. Moreover, $x$ could be an absolute point, so there remain at least $q^2 + 1 - (q + 1)$ neighbours in $x^\perp$. This shows that, for every $\epsilon > 0$, there exists a value $n_0$ such that the graph $G$ has at least

$$\frac{1}{2}(q^2 - q - 1) > \frac{1}{2}(1 - \epsilon)n^{5/3}$$

edges whenever $n \geq n_0$.

It remains to show that this graph is indeed $K_{3,3}$-free. To do so, we first show that no three vertices $x_1, x_2, x_3 \in V(G)$ can be collinear, unless they lie on a line through $z$. So suppose they are collinear, but not on a line through $z$. Then $zz_1, zz_2, zz_3$ are three distinct lines through $z$ that intersect $O$ in the points $o_1, o_2$ and $o_3$ respectively. Clearly, $\langle x_1, x_2, x_3, z \rangle$ is a plane that intersects $z^\perp$ in a line. This line contains the points $o_1, o_2, o_3 \in O$, which is in contradiction with the definition of $O$.

Now, take three vertices $x_1, x_2, x_3 \in V(G)$. Do they have at most two common neighbours? The common neighbours are points of $x_1^\perp \cap x_2^\perp \cap x_3^\perp = \langle x_1, x_2, x_3 \rangle^\perp$, so it is this subspace we have to

3. An orthogonal polarity of $PG(4, q)$ is a polarity derived from a non-degenerate symmetric bilinear form on the underlying vector space $V(5, q)$.
investigate. Suppose first that \(x_1, x_2, x_3\) are collinear. Then by the previous, \(z \in \langle x_1, x_2, x_3 \rangle\) and hence \(\langle x_1, x_2, x_3 \rangle^{+} \subseteq z^{+}\). As \(z^{+}\) contains no vertices of \(V(G)\), we conclude that \(x_1, x_2, x_3\) have no common neighbours. So assume now that \(\langle x_1, x_2, x_3 \rangle\) is a plane, then \(\langle x_1, x_2, x_3 \rangle^{+}\) is a line, not through \(z\) (as \(\langle x_1, x_2, x_3 \rangle\) is not contained in \(z^{+}\)). As we have shown above, a line not through \(z\) can contain at most two points of \(V(G)\), which concludes the claim that \(G\) is \(K_{3,3}\)-free.

This construction gives us graphs on \((q^2+1)(q-1)\) vertices for every prime power \(q\), but is this enough to prove the lower bound for all large enough natural numbers \(n\)? To do so, let \(n \in \mathbb{N}\) and find a prime power \(q\) such that \(n-r = (q^2+1)(q-1)\), where \(r\) is minimised. Then we can construct the graph on \((q^2+1)(q-1)\) vertices and add \(r\) vertices with degree zero. As \(q^3 > n-r\), the graph has at least \(\frac{1}{2}(n-r)^{5/3}\) edges. To see that, for all \(\epsilon > 0\), there exists \(n_0\), such that this is indeed greater than \(\frac{1}{2}n^{5/3}\) when \(n \geq n_0\). We need to know something about \(r\). Indeed, Bombieri’s theorem on the distribution of primes \([11]\) shows that we can choose \(r \leq c\sqrt{n}\log n\) for some constant \(c > 0\) and hence there does exist for all \(\epsilon > 0\) such an \(n_0\) (as \(\sqrt{n}\log n = o(n^{5/3})\)).

\[\square\]

**Remark 3.17.** We thank Francesco Pavese for finding a small mistake in this proof, which is due to Ball \([8]\). In the original proof, the minimum degree of a vertex is claimed to be \(q^2 + 1 - (q+1)\), which disregards the fact that \(x\) could be absolute.

First of all, the constant given here is 1/2, while Theorem 3.7 gives \(\sqrt{2}\). So it might be possible there are still better constructions (asymptotically). This is not true, as we have a sharper bound to our disposal, shown by Füredi \([38]\). The bound is originally related to the Zarankiewicz problem. The Zarankiewicz problem in full generality asks for the maximum number of 1’s in an \(m \times n\) 0-1 matrix, such that it contains no \(a \times b\) minor of 1’s. This maximum is denoted by \(Z(m,n,a,b)\). If \(m \neq n\) and \(a \neq b\), then \(Z(m,n,a,b)\) and \(Z(m,n,b,a)\) can be different, so there is some asymmetry. If \(m = n\) or \(a = b\), they are clearly equal.

**Theorem 3.18.** For all \(m \geq a, n \geq b, a \geq b \geq 2\), we have

\[Z(m,n,a,b) \leq (b-a+1)^{1/a}mn^{1-(1/a)} + (a-1)n^{2-(2/a)} + (a-2)m.\]

We can show \(2\epsilon(n, K_{a,b}) \leq Z(n,n,a,b)\) in the same way as in Chapter 2, so we immediately have following corollary, which indeed gives us the constant 1/2.

**Corollary 3.19.** \(\epsilon(n, K_{3,3}) \leq \frac{1}{2}n^{5/3} + n^{4/3} + \frac{1}{2}n\).

Secondly, if we use the original graph \(H(q, 3)\) instead of the adaptation from above, we could find that the second term in \(\epsilon(n, K_{3,3})\) is of order \(n^{4/3}\). This term was not visible in Theorem 3.7 but it is in the sharper bound, which leads us to conclude the following.

**Corollary 3.20.** \(\epsilon(n, K_{3,3}) = \frac{1}{2}n^{5/3} + \Theta(n^{4/3}).\)

Apart from \(\epsilon(n, K_{2,2})\), this is the only diagonal number \((s = t)\) for which the order of magnitude is known. Remark that when trying to prove Conjecture 3.9 it suffices to prove that \(\epsilon(n, K_{s,s}) = \Theta(n^{2-(1/s)})\) as we have for \(s \leq t\) that

\[\epsilon(n, K_{s,s}) \leq \epsilon(n, K_{s,t}),\]

since \(K_{s,s}\) is a subgraph of \(K_{s,t}\), when \(s \leq t\). Unfortunately, even for \(s = 4\), no good \(K_{4,4}\)-free graphs have been found.
3.3 Excluding $K_{2,t+1}$

As we just saw, the order of magnitude of $K_{2,t+1}$ is already determined by $\text{ex}(n, K_{2,2})$, which we saw in Chapter 2:

$$\Theta(n^{3/2}) = \text{ex}(n, K_{2,2}) \leq \text{ex}(n, K_{2,t+1}) = O(n^{3/2}).$$

We switch notation from $K_{2,t}$ to $K_{2,t+1}$, the reason is that we prefer $t+1$ and $\sqrt{t}$ instead of $t$ and $\sqrt{t-1}$.

We can show more than just the order of magnitude. Theorem 3.7 gives us a constant of $\sqrt{t}/2$. It turns out that this is the correct constant, as shown by Füredi [36].

**Theorem 3.21.** For any $t \geq 1$ we have

$$\text{ex}(n, K_{2,t+1}) = \frac{1}{2} \sqrt{t} n^{3/2} + O(n^{4/3}).$$

This is proven using finite fields and some group theory; the proof lies hence out of the scope of this thesis. This theorem was found after the corresponding Zarankiewicz number $Z(m, n, 2, t + 1)$ had been determined. Using ideas from these constructions, Füredi managed to prove the above result.

3.4 A link to designs?

Füredi makes an interesting remark at the end of the proof of the preceding result. He writes that when considering the vertices of the graph he constructs as points, and the neighbourhoods of the vertices as blocks, one nearly obtains a (symmetric) $2-(v,k,t)$ design, where $v$ and $k$ can be read out from the proof. Recall that similarly in Chapter 2, we constructed the extremal $K_{2,2}$-free graph by using projective planes, which can be viewed as (symmetric) designs, as we saw in the first section. Can we extend this idea? Given a (symmetric) design, can we construct a $K_{2,t+1}$-free graph, for some $t \geq 1$, with a high number of edges? We will try to answer these questions in the following few paragraphs.

3.4.1 Designs and $K_{2,\lambda+1}$-free graphs

As can be seen in the title, we change notation again, from $K_{2,t+1}$ to $K_{2,\lambda+1}$, as the parameter $t$ is also used in designs, but in another role. To further illustrate this: two points in a design have $\lambda$ common blocks going through them while in a $K_{2,\lambda+1}$-free graph, two vertices have at most $\lambda$ common neighbours. Switching between $\lambda$ and $t$ would get very confusing very fast, hence the change.

Starting from a design $D$, we can construct a bipartite graph with parts $P$ and $B$, where adjacency is incidence. We are not sure however, that every two blocks have at most $\lambda$ points in common. This is only guaranteed when $D$ is symmetric, as seen from Theorem 3.5. Remark that this bipartite graph construction cannot be extended for $K_{t,\lambda+1}$-free graphs when $t \geq 3$, as we again would need the design to be symmetric, but then it would be trivial as we saw in Theorem 3.4.

We can compare this situation to Chapter 2, where we could construct a $K_{2,2}$-free graph by looking at the incidence graph of a projective plane. The above construction is a generalisation of this fact. Many times, we will refer back to the previous chapter. This should not be too surprising as a projective plane is a special case of a symmetric design and many concepts and ideas we saw then will be generalised in the following few paragraphs, where possible.

3.4.2 Designs and strongly regular graphs

With an argument similar as in Chapter 2 one can find that

$$\text{ex}(n, K_{2,\lambda+1}) \leq \frac{1}{2} \sqrt{\lambda} n^{3/2} + \frac{n}{4}.$$
A link to designs?

To find this, take a $K_{2,\lambda+1}$-free graph $G$ on $n$ vertices and count triples $(x, y, z)$, where $x \sim y \sim z$, call this number $N$. On the one hand, as two vertices have at most $\lambda$ common neighbours, $N \leq n(n-1)\lambda$, with equality if and only if every two vertices have exactly $\lambda$ common neighbours. On the other hand, starting from $y$ we get

$$N = \sum_{y \in V(G)} d(y)(d(y) - 1) = \sum_{y \in V(G)} (d(y)^2 - d(y)) \geq \frac{4e^2}{n} - 2e,$$

where $e = e(G)$ and with equality if and only if $G$ is regular, by Lemma 1.6 Solving this inequality for $e$ we find

$$e \leq \frac{1}{2} \left( n - 1 \right) \frac{t}{4} + \frac{n}{4} \leq \frac{1}{2} \sqrt{n^{3/2} + \frac{n}{4}}. \quad (3.3)$$

Looking at equation (3.3), we see that the number of edges in a $K_{2,\lambda+1}$-free graph is maximised when every two vertices have $\lambda$ common neighbours and the graph is regular. These requirements give us precisely the strongly regular graphs with parameters $(v, k, \lambda, \lambda)$. In this section, we will explore the connection between these kinds of graphs and designs.

Let $G$ be a $(v, k, \lambda, \lambda)$-srg, $k \geq 2$. Then we can consider $\mathcal{P} = V(G)$ and $\mathcal{B} = \{N(x) \mid x \in V(G)\}$. The pair $(\mathcal{P}, \mathcal{B})$ fulfills axioms (D1), (D2) and (D3), and defines therefore a $2$-$(v, k, \lambda)$-design. Moreover this design is symmetric and we even have a nice bijection $\pi$ between points and blocks that maps a point $x$ to $x^\pi = N(x)$ and vice-versa. This bijection also has the property that $x \in y^\pi$ if and only if $y \in x^\pi$. This should remind you of the concept of a polarity of a projective plane.

**Definition 3.22.** A polarity of a symmetric design $D$ is a bijection $\pi$ between $\mathcal{P}$ and $\mathcal{B}$ such that $\pi^2 = \text{id}$, and for all $p \in \mathcal{P}$ and $B \in \mathcal{B}$ that $p \in B$ if and only if $B^\pi \in p^\pi$. A point $p \in \mathcal{P}$ is an absolute point of the polarity $\pi$ if $p \in p^\pi$.

Remark that in the polarity we have constructed above, no point is absolute as $x \notin N(x)$ for all vertices $x$. Such a polarity, one without any absolute points, is often called a co-null polarity. On the other hand, a polarity where every point is absolute is a null polarity. Using this terminology, we can summarize the above in following proposition.

**Proposition 3.23.** From every $(v, k, \lambda, \lambda)$-srg we can construct a symmetric $2$-$(v, k, \lambda)$-design with a co-null polarity.

Now suppose we have a symmetric $2$-$(v, k, \lambda)$-design $D$, how could we construct a strongly regular graph? First of all we would need to define what the vertices are. Do we choose $\mathcal{P}$ or $\mathcal{B}$ as the vertices? Or maybe a mix of both? Then how do we define the edges, or equivalently, the adjacency? It is not quite clear how to do this at first, but using polarities it is possible.

**Construction 3.24.** Suppose the symmetric design $D$ possesses a polarity. We can define the **polarity graph of a design** $D$ in the same way as we did in the previous chapter. Construct a graph $G$ by $V(G) = \mathcal{P}$ and two vertices $x, y$ are adjacent if and only if $x \in y^\pi$. Equivalently, $N(x) = x^\pi$ if $x$ is not absolute and $N(x) = x^\pi \setminus \{x\}$ otherwise (where we use $x$ for both the vertex and the point of the design).

We could also have constructed it by choosing the blocks to be the vertices of the graph, but the resulting graph is the same anyway. In the case that $D = PG(2, q)$, we find the polarity graph $ER_q$ again.

Clearly, if $D$ is a $2$-$(v, k, \lambda)$-design, then every vertex has degree $k$ or $k - 1$. Moreover, every two non-absolute points have $\lambda$ common neighbours. The following result is then immediate. It gives us the correspondence between $(v, k, \lambda, \lambda)$-srgs and certain symmetric $2$-$(v, k, \lambda)$-designs.
Proposition 3.25. From every symmetric 2-\((v, k, \lambda)\)-design with a co-null polarity we can construct a \((v, k, \lambda, \lambda)\)-srg.

Combining both propositions, we find following corollary.

Corollary 3.26. A \((v, k, \lambda, \lambda)\)-srg exists if and only if a symmetric 2-\((v, k, \lambda)\) design with a co-null polarity exists.

Remark 3.27. One could also have considered a correspondence between strongly regular graphs and symmetric designs with null polarities by adjusting the previous paragraph only a little. Given a strongly regular graph \(G\) and symmetric designs with null polarities by adjusting the previous paragraph only a little. Given a strongly regular graph \(G\) with parameters \((v, k - 2, \lambda)\), instead of constructing the blocks as \(N(x)\), \(x \in V(G)\), we can construct them as \(N(x) \cup \{x\}\). Conversely, given a symmetric design with null polarity, we can define vertices in the same way but adjacency as \(N(x) = x^\pi \setminus \{x\}\). In this way \((v, k - 2, \lambda)\)-srgs correspond to 2-(\(v, k + 1, \lambda)\)-designs with a null polarity. The third parameter of the strongly regular graph has to be adjusted as two vertices \(x \sim y\) will be contained (as points in the design) in \(x^\pi, y^\pi\) and \(z^\pi\) for all common neighbours \(z\) (in the graph). This should be \(\lambda\) blocks in total and hence there should be exactly \(\lambda - 2\) common neighbours.

The main observation is that in order to obtain strongly regular graphs from symmetric designs, we cannot have both absolute and non-absolute points. We prefer to keep reasoning with co-null polarities as this is a more symmetric case in the sense that both adjacent and non-adjacent vertices have the same number of common neighbours.

As an example of the strength of this correspondence we discuss the case \(\lambda = 1\). On the one hand, by the Friendship Theorem \[2.15\] we know that there exist no \((v, k, 1, 1)\)-srgs. On the other hand, every polarity of a 2-(\(v, k, 1)\)-design, which is a projective plane as we showed before, has absolute points, by Baer’s theorem \[2.11\]. Although this example is void, it is good to see that these two theorems are partially equivalent in the sense that Baer’s result implies that a \((v, k, 1, 1)\)-srg does not exist, but not that \(\tilde{F}_\lambda\) is the unique graph on \(v\) vertices such that every two vertices have one common neighbour, and that the friendship theorem implies that every polarity of a projective plane has absolute points, but does not give us the lower bound. Admittedly, the Friendship Theorem is proven originally depending on Baer’s theorem, but it can be proved independently of it, using only graph theoretical arguments. We see that both results overlap and complement each other in their own branch of combinatorics.

3.4.3 Extending the correspondence

We take a step back and try to put everything we have seen before in a bigger framework. In order to do this we consider a new class of graphs.

Definition 3.28. Let \(t \geq 2\) and \(\lambda \geq 1\). A graph on at least \(t + \lambda\) vertices has the \((t, \lambda)\)-property or is a \((t, \lambda)\)-graph if every set of \(t\) vertices have exactly \(\lambda\) common neighbours.

The reason we look at these graphs is threefold. First of all, remark that when \(t = 2\), we see that a \((v, k, \lambda, \lambda)\)-srg has the \((2, \lambda)\)-property. Finding this kind of strongly regular graphs is not so easy. Therefore, we remove the condition of regularity and try to figure out what we get. Perhaps it is impossible for \((2, \lambda)\)-graphs to be regular for certain values of \(\lambda\) (like \(\lambda = 1\)), or maybe this is a bigger class than the strongly regular graphs or something else happens. In any case, we can study properties of \((2, \lambda)\)-graphs and perhaps find as a consequence the (non-)existence of \((v, k, \lambda, \lambda)\)-srgs. A second reason is an intuitive maximality argument. In \(K_{t, \lambda+1}\)-free graphs, \(t\) vertices have at most \(\lambda\) common neighbours. Therefore, if instead of ‘at most \(\lambda\) common neighbours’ we would have ‘exactly \(\lambda\) common neighbours’, then the number of common neighbours would be the highest possible and hence the number of edges would also be maximised. This can be seen when \(t = 2\) in equation \(3.3\) (replacing the \(t\) in the equation by \(\lambda\)), where we got equality in one part if every two vertices have
going from

Let

Theorem 3.30. Let $\lambda \geq 3$, the computation becomes harder, but the intuition stays the same. Lastly, the study of $(t, \lambda)$-graphs runs quite parallel to the study of symmetric designs. In this way, we can show more connections between different subfields of combinatorics. We can classify the $(t, \lambda)$-graphs using the following few results. The first result is due to Carstens and Kruse \cite{16}, Plesník \cite{65}, and Sudolsky \cite{69} independently. The proof takes up more space than we would like, so we omit it and refer the interested reader to aforementioned references.

**Theorem 3.29.** Let $t \geq 3$, $(t, \lambda)$-graphs are precisely the complete graphs $K_{t+\lambda}$.

Therefore, $(t, \lambda)$-graphs are rather trivial when $t \geq 3$. Recall that the same phenomenon occurred in Theorem 3.4 when discussing symmetric designs with $t \geq 3$.

We are thus confined to discuss $(2, \lambda)$-graphs. We already mentioned that strongly regular graphs with parameters $(v, k, \lambda, \lambda)$ are $(2, \lambda)$-graphs, but perhaps there are more. Following proposition shows that this is not the case when $\lambda \geq 2$: the class of $(2, \lambda)$-graphs then coincides with the class of $(v, k, \lambda, \lambda)$-srgs. This result is due to Bose and Shrikhande \cite{15} and Le Conte de Poly \cite{55}.

**Theorem 3.30.** Let $\lambda \geq 2$, then a $(2, \lambda)$-graph is regular.

**Proof.** Let $G$ be a $(2, \lambda)$-graph, and take $x, y \in V(G)$. Let $d(x) = k$ and $d(y) = l$.

Suppose first that $x \not\sim y$. We will count $E(N(x), N(y))$ in two different ways. Every vertex $z \in N(x)$ has $\lambda$ common neighbours with $y$, which lead to $\lambda$ edges from $z$ to $N(y)$. However, there cannot be more edges, as then $z$ and $y$ would have more than $\lambda$ common neighbours. This gives us $k\lambda$ edges going from $N(x)$ to $N(y)$, but we have to pay attention: some have been counted twice! This happens to edges between two vertices contained in $N(x) \cap N(y)$, therefore we need to subtract this number, call it $a$. Then $E(N(x), N(y)) = k\lambda - a$. In a completely analogous way, by reversing the roles of $x$ and $y$, we see that also $E(N(x), N(y)) = l\lambda - a$ and thus $k = l$.

Now assume that $x \sim y$. Using the argument from before, we see that every vertex in $N(x)$, apart from $y$, has $\lambda$ edges going to $N(y)$ and hence $E(N(x), N(y)) = (k-1)\lambda + l - a$, where $a$ is still the number of edges between vertices in $N(x) \cap N(y)$. Similarly, with the roles of $x$ and $y$ switched, we see $E(N(x), N(y)) = (l-1)\lambda + k - a$. Combining the two equalities we find $k(\lambda - 1) = l(\lambda - 1)$ and as $\lambda \neq 1$, we find that $k = l$.

In this proof, we can also see what goes wrong for $(2, 1)$-graphs, which are the $k$-fans or windmill graphs $F_{2k+1}$: every two non-adjacent vertices indeed have the same degree, so if every vertex was not adjacent to at least one other vertex, i.e. the complement is a connected graph, this graph would also be regular. As it so happens, there is one case that is let through by this reasoning: exactly one vertex that is adjacent to all others (if we would have two, then the graph could not have the $(2, 1)$-property), this gives rise to the graph $F_{2k+1}$, as this is the only $(2, 1)$-graph containing a vertex that is adjacent to all others. The friendship theorem tells us that this is the only possible case that can occur, in particular, that regular $(2, 1)$-graphs do not exist, as we already saw.

The $(t, \lambda)$-graphs allow us to extend the correspondence between symmetric designs with a co-null polarity and strongly regular graphs. We show this by building a dictionary in Table 3.1 between the world of graphs and that of designs.
Bipartite complete graphs

| $t=2$ | $\lambda = 1$ | no $(n^2 + n + 1, n + 1, 1, 1)$-srg | $2-(n^2 + n + 1, n + 1, 1)$-design has no co-null polarities |
| $\lambda \geq 2$ | $(v, k, \lambda, \lambda)$-srg | $2-(v, k, \lambda)$-design with co-null polarity |
| $t \geq 3$ | complete graph $K_{t+\lambda}$ | $t-(t+\lambda, t+\lambda-1, \lambda)$-designs with co-null polarity |

Table 3.1: The correspondence in full force.

We can go from one side to another as we have already seen, although we haven’t yet specified the correspondence in the last. Recall from Theorem 3.4 that symmetric designs with $t \geq 3$ are exactly the $t-(v, v-1, v-t)$-designs. It is a good exercise to try and apply the ideas from previous sections to find this link. We emphasize that the left and right column correspond, but the results they contain have been discovered independently in their own subfield.

Looking at Table 3.1 it seems that the only interesting case left to explore are the $(2, \lambda)$-graphs, or equivalently, symmetric $2-(v, k, \lambda)$-designs possessing a co-null polarity. This possibility is quickly shut down by the following two theorems found respectively in [45] and [46]. Define the order of a 2-design as $n = k - \lambda$. For $\lambda = 1$, this clearly coincides with the definition of the order of a projective plane. By definition, if a design has an order, it is a 2-design.

**Theorem 3.31.** If a symmetric design of order $n$ has a co-null polarity, then $n$ is a square and $\sqrt{n}$ divides $\lambda$.

In particular when $\lambda$ is fixed, co-null polarities of symmetric $t-(v, k, \lambda)$-designs can only exist for a finite number of values of $v$. We find the same result in the graph-theoretic world.

**Theorem 3.32.** For any $\lambda \geq 2$, there are at most finitely many $(2, \lambda)$-graphs.

This means that when looking for a lower bound for $\text{ex}(n, K_{2,\lambda+1})$, we will find only finitely many values of $n$ for which $(2, \lambda)$-graphs exist and therefore these cannot give us an asymptotic lower bound.

### 3.4.4 Application: $K_{2,3}$-free graphs from biplanes

As an application of the possibilities and strengths of the correspondence we just made, we will use biplanes to construct $K_{2,3}$-free graphs with a high number of edges. Recall that a biplane is a symmetric $2-(v, k, 2)$-design. The 7 values for which biplanes are known to exist are $v = 4, 7, 11, 16, 37, 56, 79$. For the first 4 cases, we can construct polarities explicitly. For the last 3 cases, this is not so easy, therefore, we just assume that there exists a polarity (without saying anything about its number of absolute points). The orders of these biplanes are $n = 1, 2, 3, 4, 7, 9, 11$ respectively. Remark that if we know the order of the biplane, we know every parameter as $k = n + 2, v = 1 + (n + 2)(n + 1)/2$, where the last one can be seen in the following way. Fix a point $x$ of the biplane, then count pairs $(y, B)$ such that $y \in P, B \in B, x, y \in B$. This gives rise the equation

$$2(v-1) = r(k-1) = k(k-1),$$

We find the same result in the graph-theoretic world.
as \( r = k \). We will denote the number of absolute points of a polarity as \( a(\pi) \) from now on, then the number of edges in the polarity graph of a \( 2-(v, k, 2) \)-biplane is
\[
\frac{1}{2} (vk - a(\pi)).
\]

I. The biplane of order 1

This biplane is a \( 2-(4, 3, 2) \)-design. This is a trivial design \( (k = v - 1) \) and by the correspondence we discussed earlier, we know that the polarity graph is the complete graph \( K_4 \). This coincides with the fact that \( \text{ex}(n, K_{2,3}) = \binom{n}{2} \) if \( n \leq 4 \). One can see this design in a geometrical way by looking at the points (the points) and faces (the blocks) of a tetrahedron. Clearly, one can construct a polarity by mapping a point to its opposite face.

II. The biplane of order 2

The complement of the projective plane of order 2, also called the Fano plane, is a symmetric \( 2-(7, 4, 2) \)-design. By complement we mean that the points remain the points, while the blocks are obtained by taking the complement of the lines (when viewed as sets of points). In this way, one can see that a polarity of the projective plane induces a polarity of the biplane. As a polarity \( \beta \) of the Fano plane satisfies \( a(\beta) = 3 \) (this can also be found in the Baer’s paper [7]), we see that \( a(\pi) = 4 \) for a polarity \( \pi \) of the biplane.

III. The biplane of order 3

The construction of this biplane has some theoretical background in difference sets, which we will skip. The interested reader can find this material in [46]. The biplane of order 3 has 11 points and 11 lines. We will show these, together with their image under a polarity \( \pi \). One can check that the properties of a biplane and a polarity are indeed satisfied.

\[
\begin{align*}
1 & \leftrightarrow 1 \ 3 \ 4 \ 5 \ 9 & 7 & \leftrightarrow 2 \ 4 \ 5 \ 6 \ 10 \\
2 & \leftrightarrow 5 \ 7 \ 8 \ 9 \ 2 & 8 & \leftrightarrow 11 \ 2 \ 3 \ 4 \ 8 \\
3 & \leftrightarrow 8 \ 10 \ 11 \ 15 & 9 & \leftrightarrow 11 \ 1 \ 2 \ 6 \\
4 & \leftrightarrow 4 \ 6 \ 7 \ 8 \ 1 & 10 & \leftrightarrow 3 \ 5 \ 6 \ 7 \ 11 \\
5 & \leftrightarrow 10 \ 1 \ 2 \ 3 \ 7 & 11 & \leftrightarrow 6 \ 8 \ 9 \ 10 \ 3 \\
6 & \leftrightarrow 7 \ 9 \ 10 \ 11 \ 4 
\end{align*}
\]

The absolute points are marked in red. We see that for this polarity, \( a(\pi) = 5 \).

IV. The biplane of order 4

There are three biplane of order 4. One has a very nice representation. Take a \( 4 \times 4 \) grid, these are the points. The lines of the design are the points of a horizontal line and the points of vertical line, except the point in their intersection, see Figure 3.2

From this figure, we can immediately define a polarity \( \pi \) by letting each point correspond to the line they define (the horizontal and vertical line through the point, minus the point itself). This polarity has no absolute points, hence \( a(\pi) = 0 \).
Bipartite complete graphs

![Figure 3.2: The biplane of order 4.](image)

V. The biplane of order 7

Assume that this biplane has a polarity $\pi$, then we have following theorem from [45].

**Theorem 3.33.** If the order $n$ is not a square, then $v$ must be odd and $\pi$ has exactly $k$ absolute points.

Therefore, if the biplane admits a polarity, it has nine absolute points. This result also shows we cannot do better for the biplane of order 3.

VI. The biplane of order 9

The order of this biplane is a square, so we cannot apply Theorem 3.33. However, we have Theorem 3.31 to rely on. This result tells us that if the biplane of order 9 admits a polarity, it should have absolute points. Moreover, with a little more work, one can show that $a(\pi) \geq 2$.

VII. The biplane of order 11

Using Theorem 3.31 again, we see that if this biplane admits a polarity, it has $a(\pi) = 13$.

We can summarize all these results in a table for $\text{ex}(v, K_{2,3})$. In order to compare how good the lower bound is provided by the polarity, we need an explicit upper bound to compare with. This upper bound is given equation (3.3):

$$\text{ex}(v, K_{2,\lambda+1}) \leq \frac{1}{2} v \sqrt{\lambda(v - 1) + 1} + \frac{v^2}{4}.$$  

Using this upper bound, we find following table for $\text{ex}(v, K_{2,3})$ (we use $v$ instead of $n$ as $v$ denotes the number of points of the biplanes, while $n$ denotes its order).

<table>
<thead>
<tr>
<th>$v$</th>
<th>$a(\pi)$</th>
<th>lower bound</th>
<th>upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>25</td>
<td>27</td>
</tr>
<tr>
<td>16</td>
<td>0</td>
<td>48</td>
<td>48</td>
</tr>
<tr>
<td>37</td>
<td>9</td>
<td>162</td>
<td>166</td>
</tr>
<tr>
<td>$\geq 2$</td>
<td></td>
<td>$\leq 307$</td>
<td>308</td>
</tr>
<tr>
<td>79</td>
<td>13</td>
<td>507</td>
<td>513</td>
</tr>
</tbody>
</table>

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We see that in two cases, we already have equality and the other values come really close to the upper bound. Actually for \( v = 7 \) we can also show equality.

**Lemma 3.34.** \( \text{ex}(7, K_{2,3}) \leq 12 \).

*Proof.* Suppose that we have a \( K_{2,3} \)-free graph \( G \), where \( v(G) = 7 \) and \( e(G) = 13 \). We will investigate \( \Delta(G) \) and show that we can reach a contradiction in all cases. This implies \( \text{ex}(7, K_{2,3}) < 13 \).

As 
\[
\frac{7\Delta(G)}{2} \geq 13,
\]
we see that \( \Delta(G) \geq 4 \). If \( \Delta(G) = 6 \) or \( \Delta(G) = 5 \), one can take a vertex \( v \) of maximum degree and show that there exists \( w \in N(v) \) such that \( w \) has degree 3 in the subgraph induced by \( N(v) \). In particular, \( v \) and \( w \) have three common neighbours, which is a contradiction. If \( \Delta(G) = 4 \), then denote the number of vertices of degree 4 by \( a \). One can see that 
\[
\frac{4a + 3(7 - a)}{2} \geq 13,
\]
which implies that \( a \geq 5 \). In other words, we have at least five vertices of degree 4. If two vertices \( v, w \) of degree 4 were not adjacent, then they can have at most 2 common neighbours. Summing up all these vertices, we find
\[
| \{v, w\} \cup N(v) \cup N(w) | \geq 2 + 4 + 4 - 2 > 7.
\]
This implies that every two vertices of degree 4 are adjacent. As we have at least five such vertices, we see that \( G \) contains \( K_5 \), which clearly contains \( K_{2,3} \), a contradiction. \( \square \)

It remains an open question if for the other values of \( v \), these lower bounds are also extremal and if so, if the polarity graphs are the unique extremal graphs. In the case \( v = 4 \) this is clearly true, but in the case \( v = 7 \) one can construct \( W_6 \) and add a vertex adjacent to the two vertices of an edge of the pentagon \( C_5 \) in \( W_6 \). This \( K_{2,3} \)-free graph also has 12 edges and cannot be isomorphic to the polarity graph as \( 2 = \delta(G) < \Delta(G) - 1 = 4 \), which is a property of polarity graphs.

### 3.4.5 Conclusion

The moral of the story is that in the search for extremal \( K_{s,t} \)-free graphs, where every \( t \) vertices have at most \( s - 1 \) common neighbours, it might be tempting to look for graphs where every \( t \) vertices have exactly \( s - 1 \) common neighbours, as it facilitates a lot of reasoning and computations. As Theorems 3.31 and 3.32 show, our hope is in vain: the properties we ask for are too restrictive to allow for their existence. This explains some of the difficulties in finding good constructions of \( K_{s,t} \)-free graphs. Perhaps the solution lies in weakening some conditions of our ideal candidates and see what happens, like we did with \( (t, \lambda) \)-graphs, as a generalisation of \( (v, k, \lambda, \lambda) \)-sgs. We could for example weaken our definition of design: ask that through \( t \) points go at most \( \lambda \) blocks and investigate what changes. Similarly we could remove the regularity condition from strongly regular graphs and define a near-\( (v, k, \lambda, \lambda) \)-srg as a graph where every vertex has degree \( k \) or \( k - 1 \), and the rest of the properties remain. Then the polarity graph \( ER_q \) classifies as a near-\( (q^2 + q + 1, q + 1, 1, 1, 1) \)-srg. In this way, we could allow a polarity of a symmetric design to have absolute points when considering the correspondence we made. There are many avenues to be explored, but the more conditions we weaken, the harder the research gets. The key is to find the equilibrium between good combinatorial properties of the structures we consider, while their existence is still guaranteed.

**Question 3.35.** Show \( \text{ex}(n, K_{s,t}) = \Theta(n^{2-(1/s)}) \) when \( 4 \leq s \leq t < (s - 1)! + 1 \). Equivalently, determine the correct order of \( \text{ex}(n, K_{s,s}) \), when \( s \geq 4 \).

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**Question 3.36.** For any $s, t \geq 3$, find exact values of $\text{ex}(n, K_{s,t})$ for infinite values of $n$.

**Remark 3.37.** After the completion of this thesis, the author has found a new construction for $Z(m, n, 3, 2)$. We have added this in Appendix [B].
Chapter 4

Even cycles

In the previous chapter we generalized the quadrilateral $C_4 = K_{2,2}$ to complete bipartite graphs $K_{s,t}$. In this chapter, we will make the generalisation to even cycles $C_{2k}$. We already mentioned that $C_{2k}$-free graphs are hard to construct, as being $C_{2k}$-free is a global property, unlike being $K_{s,t}$-free. Secondly, in the previous chapter we had the advantage that $K_{s,t} \subseteq K_{s,t'}$ when $t \leq t'$, which helped us with determining the order of magnitude of $K_{2,t}$ and $K_{3,t}$ for example. In this chapter, $C_{2k}$ is not a subgraph of $C_{2k'}$ if $k \neq k'$, which is an extra difficulty. Therefore, results are even scarcer here than in the previous chapter. Moreover, the techniques used to prove results are more advanced than in the previous chapters. As a result, we will not be able to show the proofs here, but only refer to them.

4.0 Generalized polygons

In order to define generalized polygons, we need some graph-theoretic properties first. Let $G$ be a graph. The distance between two vertices $x, y \in V(G)$, denoted by $d(x, y)$ is the smallest $k$ such that $P_{k+1}$ is a path in $G$ with endpoints $x$ and $y$. Equivalently, it is the length of the shortest path with endpoints $x$ and $y$. For example, if $x \sim y$, then $d(x, y) = 1$ and $d(x, x) = d(y, y) = 0$. If there exists no path between two vertices then the distance is defined as $\infty$. In a connected graph, $d(x, y) \neq \infty$ for all $x, y \in V(G)$. This distance function satisfies the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z),$$

for any $x, y, z \in V(G)$. The diameter of $G$, denoted as $\text{diam}(G)$, is the longest distance between two vertices of $G$:

$$\text{diam}(G) = \max_{x,y \in V(G)} d(x, y).$$

Recall that if $G$ is not a forest, the girth of $G$ is the length of the smallest cycle of $G$. For example, the girth of $T_{n,2}$ is 4, while the girth of $C_5$ is 5.

**Definition 4.1.** Let $n \geq 2$. A point-line geometry is a generalized $n$-gon if and only if its incidence graph has diameter $n$ and girth $2n$.

Recall that the incidence graph of a point-line geometry $(P, L, I)$ is the bipartite graph with parts $P$ and $L$, and $p \in P$ is adjacent to $L \in L$ if and only if $p I L$. We call a point-line geometry a generalized polygon if it is a generalized $n$-gon for some $n \geq 2$. For example, the ordinary $n$-gon $C_n$ is an example of a generalized $n$-gon. If $S = (P, L, I)$ is a generalized polygon, then the dual $S^D = (L, P, I')$, obtained by switching the roles of points and lines (i.e. $p I L$ if and only if $L I' p$), is also a generalized polygon. Using this, when we prove a property for points, we

---

1. This implies that $P$ and $L$ are finite sets, as we only consider finite graphs.
can often translate into a property of lines and vice-versa. When \( n = 2 \), a generalized 2-gon is also called a \textbf{generalized digon}; these possess a rather trivial structure.

**Lemma 4.2.** Let \( S = (\mathcal{P}, \mathcal{L}, I) \) be a point-line geometry. Then \( S \) is a generalized digon if and only if its incidence graph is a complete bipartite graph where each part contains at least two vertices.

**Proof.** Suppose \( S \) is a generalized digon. The incidence graph is by definition bipartite, so we need to prove that every two vertices from different parts are adjacent. Suppose \( p \in \mathcal{P} \) and \( L \in \mathcal{L} \) are not adjacent, then the distance between \( p \) and \( L \) is at least three, which is in contradiction with the definition. As the incidence graph has girth four by definition, it contains a cycle of length four and therefore each part should indeed contain at least two vertices.

Now assume that the incidence graph of \( S \) is a complete bipartite graph having at least two vertices in each part. Then its diameter is 2 and its girth 4, therefore, \( S \) is a generalized digon by definition.

This means that in a generalized digon, every point is incident with every line. Fortunately, this phenomenon does not occur when \( n > 2 \).

**Lemma 4.3.** If \( n \geq 3 \), then a generalized \( n \)-gon is a partial linear space.

**Proof.** Assume otherwise, then we would find two points \( p_1, p_2 \) incident with two lines \( L_1, L_2 \). However, this would give a cycle of length four in the incidence graph, which is a contradiction.

Therefore, when \( n \geq 3 \), which is our assumption for the remainder of this chapter, we can look at the lines as sets of points and then incidence is containment. So when we have a generalized \( n \)-gon \( S \), \( n \geq 3 \), we write \( S = (\mathcal{P}, \mathcal{L}) \) instead of \( S = (\mathcal{P}, \mathcal{L}, I) \).

We see that a projective plane satisfies the definition of a generalized 3-gon, also called \textbf{generalized triangles}. One can ask if these are all the examples of generalized triangles. The answer is almost, but not quite. For example, a triangle \( K_3 \) is a generalized triangle, but does not satisfy the axiom (PP2) and hence is not a projective plane in our definition. In order to find all examples of generalized triangles, we need to adjust the definition of projective plane slightly. In particular, we change axiom (PP2) to

- (PP2'): there exist three non-collinear points.

In doing so, we obtain new point-line geometries that also classify as projective planes in the new sense. These new examples are called \textbf{degenerate projective planes}, while the projective planes satisfying the original definition are called \textbf{non-degenerate}. It is not hard to see that the degenerate projective planes are of the configuration depicted in Figure 4.1 a linear space having a line containing all points but one. In this way, a triangle \( K_3 \) is now an example of a degenerate projective plane.

**Proposition 4.4.** Let \( S = (\mathcal{P}, \mathcal{L}, I) \) be a point-line geometry. Then \( S \) is a generalized triangle if and only if \( S \) is a (possibly degenerate) projective plane.

**Proof.** Suppose \( S \) is a generalized triangle. If two points were not incident with a common line, then the distance between these two points in the incidence graph would be at least 4, which cannot be true as the diameter of the incidence graph is 3. Changing points and lines in this reasoning, we see that every two lines are incident with a point. The uniqueness of this common line and point follows from Lemma 4.3, therefore it is a linear space. As the incidence graph contains a cycle of length 6, it follows that \( S \) contains three non-collinear points. We see that it satisfies (PP1), (PP2') and is therefore a projective plane, degenerate or non-degenerate.

If \( S \) is a projective plane (in our new definition) then one can check that the incidence graph indeed has diameter 3 and girth 6. 

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In almost all examples of generalized \( n \)-gons we saw so far, there are a constant number of points on a line and a constant number of lines through a point. Only in the case of degenerate projective planes this is not true.

**Definition 4.5.** A generalized polygon has order \((s, t)\) if every line is incident with \(s + 1\) points and every point is incident with \(t + 1\) lines.

Note that if a generalized polygon \( S \) has order \((s, t)\), then \( S^D \) has order \((t, s)\). We can show that these parameters should satisfy \(s, t \geq 1\). It is immediate that the only generalized \( n \)-gons which have order \((1,1)\) are the ordinary \( n \)-gons.

**Lemma 4.6.** Let \( S \) be a generalized polygon, then every point is incident with at least two lines. Dually, every line is incident with at least two points.

**Proof.** We show that every vertex of the incidence graph is contained in a cycle of length \( 2n \) in the incidence graph, this immediately implies the lemma. Let \( x \) be an arbitrary vertex. As the incidence graph has girth \( 2n \), there exists a cycle of length \( 2n \). Take a point \( y \) in this cycle at maximal distance \( i \) of \( x \). As the diameter is \( n \), we have \( i \leq n \). Let \( z_1 \) and \( z_2 \) be the neighbours of \( y \) in the cycle, then due to the triangle inequality \( i - 1 \leq d(x, z_j) \leq i \), for \( j = 1, 2 \), which we can improve to \( d(x, z_j) = i - 1 \), \( j = 1, 2 \), because the incidence graph is bipartite. If \( i < n \), then we have a path from \( x \) to \( z_1 \) of length \( i - 1 \) and one from \( x \) to \( z_2 \) of length \( i - 1 \), which can both be extended to a path from \( x \) to \( y \). Although the two paths might share some vertices, they do not coincide and therefore we find, in any case, a cycle of length at most \( 2i < 2n \), which is a contradiction. Therefore, \( d(x, y) = n \) and \( x \) is contained in a cycle of length \( 2n \).

In particular, for every vertex \( x \) and every \( i \leq n \), there exists a vertex \( y \) such that \( d(x, y) = i \).

Although having an order appears to be a strong condition, generalized polygons already possess enough structure to satisfy this property under relatively weak conditions.

**Theorem 4.7.** Let \( S \) be a generalized \( n \)-gon, \( n \geq 3 \). If every line is incident with at least three points and every point is incident with at least three lines, then \( S \) has an order \((s, t)\).

**Proof.** As the incidence graph of \( S \) is connected, it suffices to prove that every two vertices at distance 2 have the same degree. First we show that two vertices \( v, w \) at distance \( n \) have the same degree. Consider a vertex \( x \in N(v) \), then as before, we have that \( d(w, x) = n - 1 \) and the path from \( w \) to \( x \) is unique, because the girth is \( 2n \). Therefore there is a unique \( \pi_1(x) \in N(w) \) such that \( d(x, \pi_1(x)) = n - 2 \). Similarly, for every vertex \( y \in N(w) \) we find a unique vertex \( \pi_2(y) \in N(v) \) such that \( d(y, \pi_2(y)) = n - 2 \). The maps \( \pi_1 : N(v) \to N(w) \) and \( \pi_2 : N(w) \to N(v) \) are clearly each others inverses and
therefore bijections, which gives us \( d(v) = |N(v)| = |N(w)| = d(w) \).
So in order to find that every two vertices at distance 2 have the same degree, we construct a vertex which has distance \( n \) from both. Let \( x \) and \( y \) be two vertices at distance 2 and let \( u \) be their (unique) common neighbour, then from our remark after the proof of Lemma 4.6, we know that there exists a vertex \( z \) at distance \( n - 1 \) of \( u \). One can check that indeed \( d(x, z) = d(y, z) = n \), from which the theorem follows.

From this proof one can conclude that if \( n \) is odd, then every generalized \( n \)-gon of order \((s, t)\), where \( s, t \geq 2 \), has \( s = t \): take two vertices at distance \( n \), then one of them is a point, while the other is a line. They have the same degree in the incidence graph, from which \( s = t \) follows immediately. For example when \( n = 3 \), the generalized triangles of order \((s, t)\), where \( s, t \geq 2 \), are exactly the non-degenerate projective planes, which indeed have \( s = t \), or in other words, as many points on a line as lines through a point.

The question now is for which values of \( n, s, t \) there exist generalized \( n \)-gons of order \((s, t)\) ? This is contained in the celebrated theorem of Feit and Higman [34]. A generalized \( n \)-gon for \( n = 6, 8, 12 \), is called a generalized hexagon, octagon and dodecagon respectively.

**Theorem 4.8.** Let \( S \) be a generalized \( n \)-gon, \( n \geq 3 \), of order \((s, t)\), then one of the following holds:

- \( S \) is an ordinary \( n \)-gon. In this case, \( s = t = 1 \),
- \( S \) is a non-degenerate projective plane. In this case, \( s = t \geq 2 \),
- \( S \) is a generalized quadrangle,
- \( S \) is a generalized hexagon. In this case, \( st \) is a square if \( s \neq 1 \neq t \),
- \( S \) is a generalized octagon. In this case, \( 2st \) is a square if \( s \neq 1 \neq t \),
- \( S \) is a generalized dodecagon. In this case, \( s = 1 \) or \( t = 1 \).

We come across the same phenomenon we saw in Chapter 3: when looking for combinatorial structures satisfying some conditions, their existence is often only guaranteed when the parameters of the structure are small. In this case, we are interested in generalized \( n \)-gons with order \((s, t)\), where \( s, t \geq 2 \), and these only exist when \( n = 3, 4, 6 \) or \( 8 \). In Chapter 3, we looked for symmetric \( t-(v, k, \lambda) \)-designs, but only found an infinite family when \( t = 2 \) and \( \lambda = 1 \).

In Chapter 2, when searching for \( C_4 \)-free graphs, we considered the incidence graph of a projective plane. Using a polarity of the plane we obtained a denser graph, which turned out to be extremal. As a projective plane is an example of a generalized triangle, we can try to use this approach for generalized polygons in general. The incidence graph of a generalized \( n \)-gon is \( C_{2n-2} \)-free, as its girth is \( 2n \), which is a good start. When trying to use the same approach, we will only consider generalized \( n \)-gons with an order \((s, t)\), to keep the computations feasible. By the results before, we know that these only exist when \( n = 3, 4, 6 \) or \( 8 \). To find a polarity, we need to have that \( s = t \). As the case \( n = 3 \) is already handled in Chapter 2, we will focus solely on the other possible values of \( n \). At this time, the only known generalized \( n \)-gons with order \((s, s)\), \( s \geq 2 \), are generalized quadrangles and generalized hexagons, both of order \((q, q)\), where \( q \) is a prime power. The common notations for this generalized quadrangle and hexagon are \( W(q) \) and \( H(q) \) respectively [63, 73].

It was shown by Jacques Tits, who introduced the theory of generalized polygons, that \( W(q) \) has a polarity if and only if \( q = 2^{2k+1} \), for some \( k \geq 1 \) [74]. On the other hand, \( H(q) \) has a polarity if and only if \( q = 3^{2k+1} \), for some \( k \geq 1 \), as was also shown by Tits [75]. Therefore, these are the only values for which we can construct the analogue of the polarity graph \( ER(q) \). They cannot provide an asymptotic lower bound, as the powers of 2 and 3 respectively are not dense enough among the integers. We will construct the polarity graphs anyway and count their number of edges explicitly in a later section.
4.1 Upper bounds and a probabilistic construction

In the following, we recall that \( \text{ex}(n, C_{2k}) \) is a function of \( n \), and not of \( k \). When discussing this function, we assume that \( k \geq 2 \) is a fixed integer.

The first upper bound for \( \text{ex}(n, C_{2k}) \) was stated by Erdős in 1963 [25]. He gave following upper bound, without proof:

\[
\text{ex}(n, C_{2k}) \leq c_k n^{1+1/k},
\]

where \( c_k \) is some constant only depending on \( k \). We see that for \( k = 2 \), this is indeed the correct order of magnitude. The first proof of this fact, with a more explicit constant, was given by Bondy and Simonovits [12]. They showed

\[
\text{ex}(n, C_{2k}) \leq 100 k n^{1+1/k},
\]

whenever \( n \) is large enough. This was consequently improved, first by Verstraëte, who lowered the constant \( 100 k \) to \( 8(k-1) \) [77], and then by Pikhurko [64] who showed the best current upper bound.

**Theorem 4.9.** For all \( k \geq 2 \) and \( n \geq 1 \), we have

\[
\text{ex}(n, C_{2k}) \leq (k-1)n^{1+1/k} + 16(k-1)n.
\]

Only when \( k = 2, 3, 5 \), we have constructions matching this upper bound in order of magnitude, i.e., \( \text{ex}(n, C_{2k}) = \Theta(n^{1+1/k}) \) for these values of \( k \). The first one who showed this for \( k = 3, 5 \) was Benson in 1966 [10], using generalized polygons. After him, many more constructions were found with the same order of magnitude, but using different approaches. We will discuss some of these later.

Similarly as in the previous chapter, we can construct a \( C_{2k} \)-free graph based on probabilistic methods. The order of magnitude will again be lower than the one from the preceding upper bound, but it is the only construction that can be done for arbitrary \( k \). The proof is almost a copy of the one in Chapter 3.

**Theorem 4.10.** For all \( \epsilon > 0 \), there exists \( n_0 \) such that for all \( n \geq n_0 \),

\[
\text{ex}(n, C_{2k}) > c(1-\epsilon)n^{1+1/(2k-1)},
\]

where \( c > 0 \) depends only on \( k \).

**Proof.** Again, we work in the \( G(n, p) \) model. Let \( X \) be the number of edges in \( G \). As before, we can compute its expected value:

\[
\mathbb{E}(X) = \binom{n}{2} p > cn^2 p,
\]

for any \( c < \frac{1}{2} \) if \( n \) is large enough. Similarly, if \( Y \) is the number of copies of \( C_{2k} \) in \( G \), then

\[
\mathbb{E}(Y) = \binom{n}{2k} (2k-1)!p^{2k} < n^{2k} p^{2k} / (2k),
\]

since any set of \( 2k \) vertices can be ordered to give \( (2k-1)! \) non-isomorphic cycles \( C_{2k} \). We can make \( G \) \( C_{2k} \)-free by removing an edge from every copy of \( C_{2k} \). The expected number of edges remaining is

\[
\mathbb{E}(X - Y) = \mathbb{E}(X) - \mathbb{E}(Y) > cn^2 p - n^{2k} p^{2k} / (2k).
\]

Choosing \( p = (ck)^{1/(2k-1)} n^{-1+1/(2k-1)} \), one can compute

\[
\mathbb{E}(Y - X) > c' n^{1+1/(2t-1)},
\]

for some \( c' \) which can be made explicit, but only depends on \( k \) and \( c \). If \( n \) is large enough we can choose \( c = 1/2 \) if we replace \( c' \) by \( c'(1-\epsilon) \). Hence, there exists a \( C_{2k} \)-free graph on at least the above number of edges, as we wanted. \( \square \)
2. The powers of 2 are not dense enough among the integers, so we can conclude ex(n, C₆) = Ω(n⁴/³), but not ex(n, C₆) = Θ(n⁴/³).

4.2 The polarity graph for generalized polygons

Before turning to the construction giving asymptotic equality in Theorem 4.9 we can already try and see what we find when constructing the polarity graph of generalized polygons. These were already considered by Lazebnik, Ustimenko and Woldar [53]. We saw that there exist generalized quadrangles W(q) and hexagons H(q) possessing polarities. We will denote their respective polarity graphs by PW(q) and PH(q). To find the number of edges, we can perform a similar computation as in Chapters 2 and 3. We know that both have order (q, q), so in the incidence graph, a vertex always has degree q + 1. Therefore, its degree in the polarity graph is q + 1, unless it is an absolute point, in which case the degree is q. Using this, we can compute the number of edges of the polarity graph P:

\[ e(P) = \frac{1}{2} \sum_{v \in V(P)} d(v) \]

\[ = \frac{1}{2} (v(P)(q + 1) - a), \]

where \( v(P) \) is the number of vertices of \( P \), which coincides with the number of points in the respective generalized polygon, and \( a \) is the number of absolute points of the polarity.

We will find \( v(P) \) and \( a \) both when \( P = PW(q) \) and \( P = PH(q) \), although we will only do the computations explicitly in the first case. For the second case, they are the same, although a bit more cumbersome.

We can count the number of points of \( W(q) \) and \( H(q) \) in the same way as we did in the proof of Theorem 2.41. We will do this for \( W(q) \) first. Take again an edge \( xy \) in its incidence graph and perform a breadth-first search from this edge. Denote the vertices at level \( i \geq 1 \) under \( x \) by \( N_i(x) \) and define \( N_i(y) \) analogously. Also, let \( N_0(x) = \{ x \} \) and \( N_0(y) = \{ y \} \). Then one can check that because the girth is 8, the sets \( N_i(x), N_i(y), \ i = 0, 1, 2, 3, \) are mutually disjoint, are independent sets and \( |N_i(x)| = |N_i(y)| = q^i \). Moreover, these sets contain all vertices. Suppose a vertex \( z \) is not in any of these sets, then \( z \) has distance at least 4 to either \( x \) or \( y \) and hence distance at least 5 to \( y \) or \( x \) respectively (as the incidence graph is bipartite), which is in contradiction with the definition of a generalized quadrangle. In this way, we can count the number of vertices of \( PW(q) \) easily:

\[ V(PW(q)) = \bigcup_{i=0}^{3} (N_i(x) \cup N_i(y)) \]

\[ \Rightarrow v(PW(q)) = 2 + 2q + 2q^2 + 2q^3. \]

Clearly, we have as many points as lines, so \( |P(W(q))| = |L(W(q))| = q^3 + q^2 + q + 1 \). We already see that the incidence graph is \( C_6 \)-free, has approximately \( 2q^3 \) vertices, \( q^4 \) edges (asymptotically speaking) and therefore gives us the correct order of magnitude (up to constant \( c \), \( \text{ex}(n, C_6) \approx cn^{4/3} \) for infinitely many values of \( n \), i.e. when \( n = 2^{2k+1}, k \geq 1 \))

The same occurred when looking at the incidence graph of a projective plane.

Using the same technique we see that

\[ V(PH(q)) = \bigcup_{i=0}^{4} (N_i(x) \cup N_i(y)) \]

\[ \Rightarrow v(PH(q)) = 2 + 2q + 2q^2 + 2q^3 + 2q^4 + 2q^5, \]

and therefore \( |P(H(q))| = |L(H(q))| = q^5 + q^4 + q^3 + q^2 + q + 1 \).

The only thing remaining is finding the number of absolute points \( a \) for a polarity of \( W(q) \) and \( H(q) \) respectively. We will again compute it explicitly only for \( W(q) \), as the argument for \( H(q) \) is exactly the same.
same, only a bit more complicated. In order to find \(a\) for \(W(q)\), we will need the concept of ovoid of a
generalized quadrangle. Recall that we already came across ovoids in Chapter 3, these were ovoids of
\(PG(3, q)\). The definition here is of a different kind.

**Definition 4.11.** An ovoid of a generalized quadrangle is a set of points \(O\) such that every line
contains exactly one point of \(O\).

The reason we introduce these objects, are immediately clear by the following result.

**Proposition 4.12.** If a generalized quadrangle \(S\) has a polarity \(\theta\), then the set of its absolute points form
an ovoid of \(S\).

In the proof we use following argument: if a point \(x\) in a generalized quadrangle is not contained in
a line \(L\), then \(x\) is collinear with exactly one point of \(L\). This follows as the diameter of the incidence
graph of a generalized quadrangle is 4, so \(x\) is collinear with at least one point of \(L\). As the girth is 8,
it follows that this point is unique.

**Proof.** First we prove that no line can contain more than one absolute point. Suppose that \(x, y\) are
absolute points on a line \(L\). Then \(x \in x^\theta, y \in y^\theta\) and \(L^\theta = (xy)^\theta = x^\theta \cap y^\theta\). If \(L^\theta \notin L\), then \(L^\theta, x, y\)
would be a triangle, which is impossible in a generalized quadrangle. Therefore \(L^\theta \in L\) and hence
\(L^\theta \in L \cap x^\theta \cap y^\theta\), which is impossible as \(x\) and \(y\) are two different absolute points.
If \(L\) is an absolute line, then \(L^\theta \in L\) is an absolute point, and by the above we know that this is the
unique absolute point on \(L\). So suppose \(L\) is not absolute. Then the point \(L^\theta\) is not on the line \(L\), so
is collinear with a point \(u \in L\), denote this line as \(M = L^\theta u\). Then \(u = L \cap M\), so \(L^\theta M^\theta = u^\theta\).
Also, \(L^\theta \in M\), hence \(M^\theta \in L\). We see that \(L^\theta M^\theta\) is a line from \(L^\theta\) to a collinear point \(M^\theta\) on \(L\). By
uniqueness, it follows that \(u = M^\theta\) and \(u \in M = u^\theta\). Then \(u\) is the unique absolute point on \(L\).

Reducing the problem of finding the number of absolute points to finding the number of points of
ovoids immediately gives us the answer.

**Lemma 4.13.** An ovoid \(O\) of \(W(q)\) has \(q^2 + 1\) points.

**Proof.** We double count point-line pairs \((p, L)\), where \(p \in L \cap O\). For each point of the ovoid, we find
\(q + 1\) lines through the point. When starting from a line, we have by definition that the line contains
exactly one point of \(O\). Combining this we see

\[ |O|(q + 1) = q^3 + q^2 + q + 1 = (q + 1)(q^2 + 1). \]

Combining all arguments we see that

\[
\begin{align*}
v(PW(q)) &= q^3 + q^2 + q + 1, \\
e(PW(q)) &= \frac{1}{2}(q^3 + q^2 + q + 1)(q + 1) - (q^2 + 1) = \frac{1}{2}q(q + 2)(q^2 + 1),
\end{align*}
\]

which is indeed of the correct order of magnitude. The same results hold for \(H(q)\): the set of absolute
points form an ovoid (which has a slightly different definition for generalized hexagons) and the
number of points on an ovoid of \(H(q)\) is \(q^3 + 1\). Therefore we get

\[
\begin{align*}
v(PH(q)) &= q^5 + q^4 + q^3 + q^2 + q + 1, \\
e(PH(q)) &= \frac{1}{2}((q^5 + q^4 + q^3 + q^2 + q + 1)(q + 1) - (q^3 + 1)) = \frac{1}{2}q(q + 1)(q^4 + q^3 + 2),
\end{align*}
\]

which also has the correct order of magnitude. In fact, Erdős and Simonovits \cite{31} conjectured that
Conjecture 4.14. For all $k \geq 2$,

$$\text{ex}(n, C_{2k}) = \frac{1}{2}n^{1+1/k} + o(n^{1+1/k}).$$

If this conjecture were true, then $PW(q)$ and $PH(q)$ would even give the correct constant for $k = 3$ and $k = 5$ respectively and have a big chance to be extremal, as we found for $k = 2$ with $ER(q)$. Unfortunately, in both cases the answer is negative, as we will see in the next section.

### 4.3 Better $C_{2k}$-free graphs

After Benson’s constructions, 20 years passed without new contributions until Wenger [78] constructed $C_{2k}$ free graphs for $k = 2, 3, 5$ using algebraic methods. These graphs were generalized by Lazebnik and Ustimenko [51] using group theory and algebra. They showed that the $C_6$-free graph constructed by Wenger is actually an induced subgraph of the incidence graph of a generalized quadrangle, while his $C_{10}$-free graph is not a subgraph of a generalized hexagon. Finally, Mellinger and Mubayi [58] provided new constructions for $C_6$- and $C_{10}$-free graphs. We will discuss these more in depth.

Their idea is a generalisation of the construction by Mellinger [57], who constructed a $C_6$-free graph in the following way. First, we need the definition of an oval of $PG(2, q)$.

**Definition 4.15.** An oval $O$ is a set of $q + 1$ points of $PG(2, q)$ such that no three are collinear.

We already came across this concept when considering the absolute points of an orthogonal polarity of $PG(2, q)$. These indeed form an oval.

We construct a graph in the following way. Let $q$ be a prime power and consider $PG(3, q)$ and a plane $H$ in $PG(3, q)$. In this plane, we take an oval $O$. Recall that this is a set of $q + 1$ points such that no three of which are collinear. Then we construct the bipartite $C_6$-free graph $G_O$ in the following way. The vertices are the points of $PG(3, q) \setminus H$ and the lines of $PG(3, q)$ through a point of $O$, but not contained in $H$. Adjacency is defined as containment.

**Proposition 4.16.** The graph $G_O$ contains no $C_6$.

**Proof.** Suppose, for the sake of contradiction, that we find $C_6$ as a subgraph. Then in the projective space $PG(3, q)$, this would correspond with three points $x, y, z \in PG(3, q) \setminus H$ such that the lines $xy, xz, yz$ intersect $H$ in a point of $O$, say $o_1, o_2$ and $o_3$ respectively. These three lines are contained in a plane $\pi = \langle x, y, z \rangle$, which is distinct from $H$. Therefore, the intersection of $\pi$ and $H$ is a line containing $o_1, o_2$ and $o_3$, which is clearly a contradiction. \hfill $\Box$

Actually when $q$ is even, the graph $G_O$ is a subgraph of the incidence graph of a generalized quadrangle of order $(q - 1, q + 1)$. It seems that the authors were unaware of this fact. Therefore, this seemingly new construction again has his roots in the theory of generalized polygons. The advantage of this point of view however is that they managed to construct $C_{10}$-free graphs, which do not come from generalized hexagons. In order to do so, they consider $PG(5, q)$, a hyperplane $H$ and a set $A$ of $q + 1$ points in $H$ such that no 5 are contained in a three-dimensional space (this set exists when $q \geq 5$, see for example [43]). Then consider a bipartite graph where the vertices are the points of $PG(5, q) \setminus H$ and the lines through a point of $A$, not contained in $H$. Doing a case-by-case check (which we will omit), they prove that this graph contains no $C_{10}$-free. All of these constructions show that, for $k = 2, 3, 5$,

$$\text{ex}(n, C_{2k}) \geq \frac{1}{2}n^{1+1/k}.$$
Even cycles

Note that now, we can write this inequality for general \( n \), as the graphs can be constructed for all prime powers \( q \), which are dense enough among the integers. These constructions all support Conjecture 4.14. However, in 1994, Lazebnik, Ustimenko and Woldar [52] managed to prove that, for \( k = 3, 5 \),

\[
\text{ex}(n, C_{2k}) \geq \frac{k - 1}{k^{1+1/k}} n^{1+1/k} + o(n^{1+1/k}).
\]

When \( k = 5 \), this improves the constant from \( 1/2 \) to \( 4/5^{6/5} \approx 0.58 \) and disproves the conjecture. For \( k = 3 \) however, this improvement was not strong enough as \( 2/3^{4/3} \approx 0.46 \). Finally, in 2006, Füredi, Naor and Verstraëte [40] managed to disprove the conjecture in the remaining case \( k = 3 \). They managed to show that

\[
\text{ex}(n, C_6) > 0.53n^{4/3}.
\]

The conjecture is still wide open for all other values of \( k \), not even the magnitude of \( \text{ex}(n, C_{2k}) \) is known when \( k = 4 \). A clear approach towards solving this problem remains absent.

**Question 4.17.** Determine \( c_k \) such that \( \text{ex}(n, C_{2k}) = c_k n^{1+1/k} + o(n^{1+1/k}) \) for \( k = 3, 5 \). Prove or disprove that \( \text{ex}(n, C_8) = \Theta(n^{5/4}) \), or more generally that \( \text{ex}(n, C_{2k}) = \Theta(n^{1+1/k}) \).
Appendix A

Samenvatting

We bestuderen een probleem in extremale grafentheorie en onderzoeken de link met eindige meetkunde. Het deelgebied van dit domein waar we ons op richten is dat van de bepaling van Turán-getallen. Dit gebied is genoemd naar Pál Turán, die in 1941 bepaalde wat het maximaal aantal bogen is in een graaf op $n$ toppen die geen complete graaf $K_r$ bevat. Dit resultaat is het ontstaan van een nieuwe tak van de extremale grafentheorie die nu Turán-problemen wordt genoemd. Er is een grote algemeenheid waarin men dit probleem kan onderzoeken, maar wij focussen ons op de vraag: gegeven een graaf $G$ op $n$ toppen, wat is het maximaal aantal bogen deze graaf kan hebben zonder een bepaalde deelgraaf $H$ te bevatten? Dit maximum wordt genoteerd als $ex(n, H)$, en bestudeerd als een functie in $n$. Na de bepaling van dit maximum wordt er gekeken welke grafen dit maximum bereiken, deze verzameling van extremale grafen wordt genoteerd als $EX(n, H)$. Erdős, Stone en Simonovits bepaalden de asymptotische groei van $ex(n, H)$ in het geval $H$ niet bipartiet is, dit wordt het niet-ontaarde geval genoemd. In het ontaarde geval, wanneer de verboden subgraaf $H$ bipartiet is, is er slechts weinig geweten.

In hoofdstuk 1 onderzoeken we de functie $ex(n, H)$ voor arbitraire $H$ en bekijken een paar specifieke gevallen van $H$ waarin de waarden van deze functie exact bekend zijn. In hoofdstuk 2 bestuderen we het geval dat $H$ de vierhoek $C_4 = K_{2,2}$ is. Voor bepaalde waarden van $n$ zijn $ex(n, C_4)$ en $EX(n, C_4)$ bekend. Voor deze waarden spelen eindige projectieve vlakken een belangrijke rol. Zij komen op natuurlijke manier naar voor wanneer men constructies van $C_4$-vrije grafen bestudeert. In hoofdstuk 3 proberen we deze resultaten te veralgemenen naar algemene compleet bipartiete grafen $K_{s,t}$. We zien dat hier designs een belangrijke rol spelen. Niet toevallig is een projectief vlak een voorbeeld van een design. Ten slotte, in hoofdstuk 4, kijken we naar het geval de verboden subgraaf een even cykel $C_{2k}$ is. In dit geval is er nog zeer weinig gekend. De meetkundige structuren die hier aanleiding geven tot $C_{2k}$-vrije grafen zijn de veralgemeende veelhoeken. Ook hier zijn de projectieve vlakken een deelklasse van. Op deze manier zien we dat in hoofdstuk 3 en 4 zowel de graaf als de eindige meetkunde die op natuurlijke wijze er bij verschijnt, veralgemeend worden. Deze connectie is het uitgangspunt van de thesis.
Appendix B

New construction for $Z(m, n, 2, 3)$

We recall the upper bound for the Zarankiewicz problem by Füredi [38].

**Theorem B.1.** For all $m \geq a, n \geq b, b \geq a \geq 2$, we have

$$Z(m, n, a, b) \leq (b - a + 1)^{1/a} mn^{1-(1/a)} + (a - 1)n^{2-(2/a)} + (a - 2)m.$$ 

We will construct a bipartite graph with classes $V_1$ and $V_2$ of size $q^2 + q$ and $q^3$ respectively, for every odd prime power $q$. We will show that every two vertices of $V_1$ have at most two common neighbours. Also recall the definition of an oval. From now on we assume that $q$ is odd.

**Definition B.2.** An oval $O$ of $PG(2, q)$ is a set of $q + 1$ points such that no three are collinear.

In Chapter 2, we proved that through every point of $O$, there is a unique tangent line intersecting $O$ only in the point. Moreover, every point not in $O$ were on 0 or 2 tangent lines. This will lead to following construction.

**Construction B.3.** Consider $PG(3, q)$ and a hyperplane $H = PG(2, q)$ in this space. In $H$, consider an oval $O$. Let $V_1$ be the set of points of $PG(3, q)$, not in $H$. Let $V_2$ be the set of planes, through a tangent line of $O$, but not equal to $H$. Then define the bipartite graph $G$ on classes $V_1$ and $V_2$ where adjacency is containment.

**Proposition B.4.** Two vertices in $V_1$ have at most two common neighbours.

**Proof.** Take two vertices $x, y \in V_1$. These are two points in $PG(3, q)$, but not in $H$, so the line $L = xy$ intersects $H$ in a point $z$. Through this point $z$ there are at most two tangent lines $L_1, L_2$. The planes $\langle L, L_1 \rangle, \langle L, L_2 \rangle$ are the only common neighbours of $x$ and $y$.

We have to make two remarks about this construction. First, when $q$ is even, it can be shown that there exists a unique point incident with all tangent lines. Therefore, if $L$ intersects $H$ in this point, there would be $q + 1$ planes adjacent with $x$ and $y$, which is not what we want. Therefore, the construction can only be done when $q$ is odd.

Secondly, the graph is not $K_{2,3}$-free: take any two planes $\pi_1, \pi_2$, then $\pi_1 \cap \pi_2$ contains $q \geq 3$ points of $PG(3, q)$, not in $H$. Therefore, it is important that we consider the asymmetric problem by Zarankiewicz, rather than the Turán problem.

**Corollary B.5.** For all odd prime powers $q$, we have

$$Z(q^3, q^2 + q, 2, 3) = \Theta(q^4)$$.
Proof. The upper bound follows from Theorem B.1. The lower bound comes from our construction.
Every vertex in $V_1$ has degree $q + 1$: a point in $PG(3, q)$, not in $H$, together with any of the $q + 1$
tangent lines, determines a plane of $V_2$. Therefore,

$$q^3(q + 1) \leq Z(q^3, q^2 + q, 2, 3).$$

We have seen these types of constructions before: first taking a projective space and a hyperplane
herein, then taking an appropriate set of points in the hyperplane. A bipartite graph is constructed
where the vertices are the points not in the hyperplane and the lines intersecting the hyperplane in
a point of the chosen set. The main novelty in this construction is not that it shows the preceding
result, but in the fact that it shows the first of its kind (as far as we aware): it uses points and planes instead
of the usual points and lines. This shows that these types of constructions are very powerful in their
flexibility. One can adapt them to investigate other extremal problems. The disadvantage of these
types of constructions is while they are able to give asymptotic results, it is doubtful whether these
provide extremal graphs.
Bibliography


